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Handout 19 Principles of Digital Communications
Problem Set 8 Apr. 22, 2015

PROBLEM 1. *(Suboptimal Receiver for Orthogonal Signaling)* This exercise takes a different approach to the evaluation of the performance of Block-Orthogonal Signaling (Example 4.6). Let the message $H \in \{1, \dots, m\}$ be uniformly distributed and consider the communication problem described by:

$$H = i : \qquad Y = c_i + Z, \quad Z \sim \mathcal{N}(0, \sigma^2 I_m),$$

where $Y = (Y_1, \dots, Y_m)^\mathsf{T} \in \mathbb{R}^m$ is the received vector and $\{c_1, \dots, c_m\} \subset \mathbb{R}^m$ the codebook consisting of constant-energy codewords that are orthogonal to each other. Without loss of essential generality, we can assume

$$c_i = \sqrt{\mathcal{E}} e_i,$$

where e_i is the i-th unit vector in $\mathbb{R}^m$, i.e., the vector that contains 1 at position i and 0 elsewhere, and $\mathcal{E}$ is some positive constant.

(a) Describe the statistic of Y_j for $j = 1, \dots, m$ given that $H = 1$.

(b) Consider a suboptimal receiver that uses a threshold $t = \alpha\sqrt{\mathcal{E}}$ where $0 < \alpha < 1$. The receiver declares $\hat{H} = i$ if i is the *only* index such that $Y_i \geq t$. If there is no such i or there is more than one index i for which $Y_i \geq t$, the receiver declares that it cannot decide. This will be viewed as an error. Let $E_i = \{Y_i \geq t\}$, $E_i^c = \{Y_i < t\}$, and describe, in words, the meaning of the event

$$E_1 \cap E_2^c \cap E_3^c \cap \cdots \cap E_m^c.$$

(c) Find an upper bound to the probability that the above event *does not* occur when $H = 1$. Express your result using the Q function.

(d) Now let $m = 2^k$ and $\mathcal{E} = k\mathcal{E}_b$ for some fixed energy-per-bit $\mathcal{E}_b$. Prove that the error probability goes to 0 as $k \to \infty$ provided that

$$\frac{\mathcal{E}_b}{\sigma^2} > \frac{2 \ln 2}{\alpha^2}.$$

Hint: Use $m - 1 < m = \exp(\ln m)$ and $Q(x) < \frac{1}{2}\exp(-\frac{x^2}{2})$.

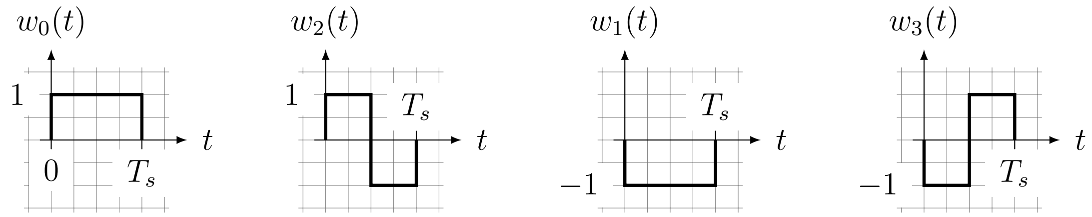
Notice that because we can choose α^2 as close to 1 as desired, if we insert $\sigma^2 = \frac{N_0}{2}$ the condition becomes $\frac{\mathcal{E}_b}{N_0} > \ln 2$, which is a weaker condition than the one obtained in Example 4.6.

Problem 2. *(Bit-By-Bit on a Pulse Train)* A communication system uses bit-by-bit on a pulse train to communicate at 1 Mbps using a rectangular pulse. The transmitted signal is of the form

$$\sum_j B_j \mathbb{1}\{t \in [jT_s, (j+1)T_s)\},$$

where $B_j \in \{\pm b\}$. Determine the value of b needed to achieve bit-error probability $P_b = 10^{-5}$ knowing that the channel corrupts the transmitted signal with additive white Gaussian noise of power spectral density $N_0/2$ where $N_0 = 10^{-2}$ Watts/Hz.

Problem 3. *(Bit Error Probability)* A discrete memoryless source produces bits at a rate of 10^6 bps. The bits, which are uniformly distributed and i.i.d., are grouped into pairs and each pair is mapped into a distinct waveform and sent over an AWGN channel of noise power spectral density $N_0/2$. Specifically, the first two bits are mapped into one of the four waveforms shown in the figure below with $T_s = 2 \times 10^{-6}$ seconds, the next two bits are mapped onto the same set of waveforms delayed by T_s, etc.



(a) Describe an orthonormal basis for the inner product space $\mathcal{W}$ spanned by $w_i(t)$, $i = 0, \ldots, 3$ and plot the signal constellation in $\mathbb{R}^n$, where n is the dimensionality of $\mathcal{W}$.

(b) Determine an assignment between pairs of bits and waveforms such that the bit error probability is minimized and derive an expression for P_b.

(c) Draw a block diagram of the receiver that achieves the above P_b and uses a single and causal filter.

(d) Determine the energy per bit $\mathcal{E}_b$ and the power of the transmitted signal.

Problem 4. *(m-ary Frequency Shift Keying)* m-ary Frequency Shift Keying (m-FSK) is a signaling method that uses signals of the form

$$w_i(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cos\Big(2\pi(f_c + i\Delta f)t\Big) \mathbb{1}\{t \in [0, T]\}, \quad i = 0, \cdots, m-1,$$

where $\mathcal{E}$, T, f_c, and Δf are fixed parameters with $\Delta f \ll f_c$.

(a) Determine the average energy assuming $f_c T$ is an integer.

(b) Assuming that $f_c T$ is an integer, find the smallest value of Δf that makes $w_i(t)$ orthogonal to $w_j(t)$ when $i \neq j$.

(c) In practice the signals $w_i(t), \quad i = 0, 1, \cdots, m-1$ can be generated by changing the frequency of a signal oscillator. In passing from one frequency to another a phase shift θ is introduced. Again, assuming that $f_c T$ is an integer, determine the smallest value Δf that ensures orthogonality between $\cos(2\pi(f_c + i\Delta f)t + \theta_i)$ and $\cos(2\pi(f_c + j\Delta f)t + \theta_j)$ whenever $i \neq j$ regardless of θ_i and θ_j.

(d) Sometimes we do not have complete control over f_c either, in which case it is not possible to set $f_c T$ to an integer. Argue that if we choose $f_c T \gg 1$ then for all practical purposes the signals will be orthogonal to one another.

(e) Determine the approximate frequency-domain interval occupied by the signal constellation. How does the WT product behave as a function of $k = \log_2(m)$?

Problem 5. *(Antipodal Signaling and Rayleigh Fading)* Consider using antipodal signaling, i.e. , $w_0(t) = -w_1(t)$, to communicate one bit across a Rayleigh fading channel that we model as follows. When $w_i(t)$ is transmitted, the channel output is

$$R(t) = Aw_i(t) + N(t),$$

where $N(t)$ is white Gaussian noise of power spectral density $N_0/2$ and A is a random variable of probability density function

$$f_A(a) = \begin{cases} 2ae^{-a^2}, & \text{if } a \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We assume that, unlike the transmitter, the receiver knows the realization of A. We also assume that the receiver implements a maximum likelihood decision and the signal's energy is $\mathcal{E}_b$.

(a) Describe the receiver.

(b) Determine the error probability conditioned on the event $\{A = a\}$.

(c) Determine the unconditional error probability P_f (the subscript stands for fading).

(d) Compare P_f to the error probability P_e achieved by an ML receiver that observes $R(t) = mw_i(t) + N(t)$ where $m = \mathbb{E}[A]$. Comment on the different behavior of the two error probabilities. For each of them find the $\mathcal{E}_b/N_0$ value necessary to obtain the probability of error 10^{-5}.

Problem 6. *(Non-White Gaussian Noise)* Consider the following transmitter/receiver design problem for an additive non-white Gaussian noise channel.

(a) Let the hypothesis H be uniformly distributed in $\mathcal{H} = \{0, \ldots, m-1\}$ and when $H = i,\ i \in \mathcal{H}$, let $w_i(t)$ be the channel input. The channel output is then

$$R(t) = w_i(t) + N(t)$$

where $N(t)$ is a colored Gaussian noise of power spectral density $G(f)$ where we assume $G(f) \neq 0$ for all f. Describe a receiver that, based on the channel output $R(t)$, decides on the value of H with least probability of error.

Hint: Find a way to transform this problem into one that you can solve.

(b) Consider the setting in part (a) except that now you get to design the signal set with the restrictions that $m = 2$ and that the average energy cannot exceed $\mathcal{E}$. We also assume that $G(f)$ is constant in the interval $[a, b]$, $a < b$, where it also achieves its global minimum. Find two signals that achieve the smallest possible error probability under the ML decoding rule.