ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 12 Principles of Digital Communications

Problem Set 6 Mar. 25, 2015

PROBLEM 1. *(On-Off Signaling)* Consider the binary hypothesis testing problem specified by:

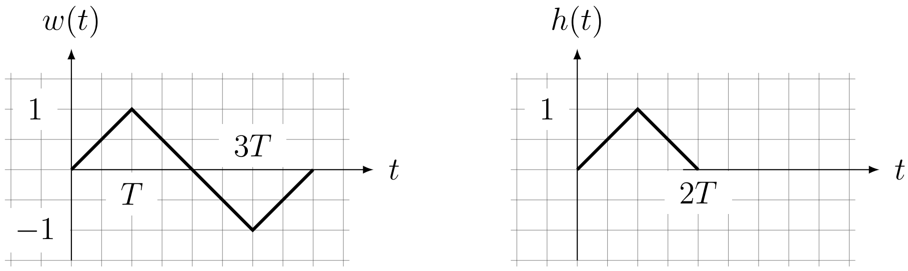
$$\begin{aligned} H = 0 &\ :\ R(t) = w(t) + N(t) \\ H = 1 &\ :\ R(t) = N(t) \end{aligned}$$

where $N(t)$ is additive white Gaussian noise of power spectral density $N_0/2$ and $w(t)$ is the signal shown in the left figure.

(a) Describe the maximum likelihood receiver for the received signal $R(t)$, $t \in \mathbb{R}$.

(b) Determine the error probability for the receiver you described in (a).

(c) Sketch a block diagram of your receiver of part (a) using a filter with impulse response $h(t)$ (or a scaled version thereof) shown in the right figure.



PROBLEM 2. *(QAM Receiver)* Let the channel output be

$$R(t) = W(t) + N(t),$$

where $W(t)$ has the form

$$W(t) = \begin{cases} X_1 \sqrt{\frac{2}{T}} \cos 2\pi f_c t + X_2 \sqrt{\frac{2}{T}} \sin 2\pi f_c t, & \text{for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases}$$

$2f_cT \in \mathbb{Z}$ is a constant known to the receiver, $X = (X_1, X_2)$ is a uniformly distributed random vector that takes values in

$$\{(\sqrt{\mathcal{E}}, \sqrt{\mathcal{E}}), (-\sqrt{\mathcal{E}}, \sqrt{\mathcal{E}}), (-\sqrt{\mathcal{E}}, -\sqrt{\mathcal{E}}), (\sqrt{\mathcal{E}}, -\sqrt{\mathcal{E}})\}$$

for some known constant $\mathcal{E}$, and $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$.

(a) Specify a receiver that, based on the channel output $R(t)$, decides on the value of the vector X with least probability of error.

(b) Find the probability of error of the receiver you have specified.

PROBLEM 3. *(Matched Filter Implementation)* In this problem, we consider the implementation of matched filter receivers. In particular, we consider Frequency Shift Keying (FSK) with the following signals:

$$w_j(t) = \begin{cases} \sqrt{\frac{2}{T}} \cos 2\pi \frac{n_j}{T} t, & \text{for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases}$$

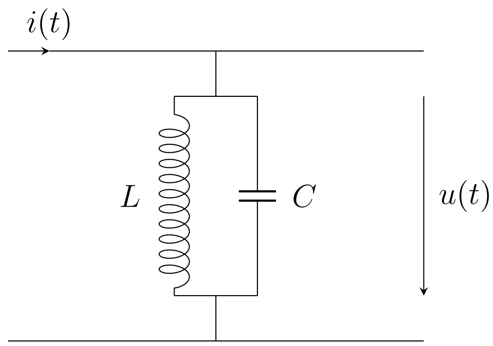
where $n_j \in \mathbb{Z}$ and $0 \le j \le m-1$. Thus, the communication scheme consists of m signals $w_j(t)$ of different frequencies $\frac{n_j}{T}$.

(a) Determine the impulse response $h_j(t)$ of a causal matched filter for the signal $w_j(t)$. Plot $h_j(t)$ and specify the sampling time.

(b) Sketch the matched filter receiver. How many matched filters are needed?

(c) Sketch the output of the matched filter with impulse response $h_j(t)$ when the input is $w_j(t)$.

(d) Consider the following ideal resonance circuit:



For this circuit, the voltage response to the input current $i(t) = \delta(t)$ is

$$h(t) = \begin{cases} \frac{1}{C} \cos \frac{t}{\sqrt{LC}}, & t \ge 0, \\ 0 & t < 0. \end{cases}$$

Show how this can be used to implement the matched filter for signal $w_j(t)$. Determine how L and C should be chosen.

Hint: Suppose that $i(t) = w_j(t)$. In this case, what is $u(t)$?

Problem 4. *(Matched Filter Intuition)* In this problem, we develop some further intuition about matched filters. You may assume that all waveforms are real valued. Let

$$R(t) = \pm w(t) + N(t)$$

be the channel output, where $N(t)$ is additive white Gaussian noise of power spectral density $N_0/2$ and $w(t)$ is an arbitrary but fixed waveform. Let $\phi(t)$ be a unit-norm but otherwise arbitrary waveform and consider the receiver operation

$$Y = \langle R, \phi \rangle = \langle w, \phi \rangle + \langle N, \phi \rangle$$

The signal-to-noise ratio (SNR) is thus

$$SNR = \frac{|\langle w, \phi \rangle|^2}{\mathbb{E}\left[|\langle N, \phi \rangle|^2\right]}.$$

Notice that the SNR is not changed when $\phi(t)$ is multiplied by a constant. Notice also that

$$\mathbb{E}\left[|\langle N, \phi \rangle|^2\right] = \frac{N_0}{2}.$$

(a) Use the Cauchy-Schwarz inequality to give an upper bound on the SNR. What is the condition for equality in the Cauchy-Schwarz inequality? Find the $\phi(t)$ that maximizes the SNR. What is the relationship between the maximizing $\phi(t)$ and the signal $w(t)$?

(b) Let us verify that we would get the same result using a pedestrian approach. Instead of waveforms, consider tuples. Let $c = (c_1, c_2)^\mathsf{T}$ and use calculus (instead of the Cauchy-Schwarz inequality) to find the $\phi = (\phi_1, \phi_2)^\mathsf{T}$ that maximizes $\langle c, \phi \rangle$ subject to the constraint that ϕ has unit energy.

(c) Verify with a picture (convolution) that the output at time T of a filter with input $w(t)$ and impulse response $h(t) = w(T - t)$ is indeed $\|w\|^2 = \int w^2(t)\, dt$.

Problem 5. *(AWGN Channel and Sufficient Statistic)* Let $\mathcal{W} = \{w_0(t), w_1(t)\}$ be the signal constellation used to communicate an equiprobable bit across an additive Gaussian noise channel. In this problem, we verify that the projection of the channel output onto the inner-product space $\mathcal{V}$ spanned by $\mathcal{W}$ is not necessarily a sufficient statistic, unless the noise is white. Let $\psi_1(t), \psi_2(t)$ be an orthonormal basis for $\mathcal{V}$. We choose the additive noise to be $N(t) = Z_1\psi_1(t) + Z_2\psi_2(t) + Z_3\psi_3(t)$ for some normalized $\psi_3(t)$ that is orthogonal to $\psi_1(t)$ and $\psi_2(t)$ and choose Z_1, Z_2 and Z_3 to be zero-mean jointly Gaussian random variables of identical variance σ^2. Let $c_i = (c_{i,1}, c_{i,2}, 0)^\mathsf{T}$ be the codeword associated to $w_i(t)$ with respect to the extended orthonormal basis $\psi_1(t), \psi_2(t), \psi_3(t)$. There is a one-to-one correspondence between the channel output $R(t)$ and $Y = (Y_1, Y_2, Y_3)^\mathsf{T}$ where $Y_i = \langle R, \psi_i \rangle$. In terms of Y, the hypothesis testing problem is

$$H = i \; : \; Y = c_i + Z \quad i = 0, 1$$

where we have defined $Z = (Z_1, Z_2, Z_3)^\mathsf{T}$.

(a) As a warm-up exercise, let us first assume that Z_1, Z_2 and Z_3 are independent. Use the Fisher–Neyman factorization theorem to show that (Y_1, Y_2) is a sufficient statistic.

(b) Now assume that Z_1 and Z_2 are independent but $Z_3 = Z_2$. Prove that in this case (Y_1, Y_2) is *not* a sufficient statistic.

(c) To check a specific case, consider $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$. Determine the error probability of an ML receiver when it observes $(Y_1, Y_2)^\mathsf{T}$ and that of another ML receiver that observes $(Y_1, Y_2, Y_3)^\mathsf{T}$.

PROBLEM 6. *(Mismatched Receiver)* Let the channel output be

$$R(t) \quad = \quad c\, X\, w(t) + N(t), \tag{1}$$

where $c > 0$ is some deterministic constant, X is a uniformly distributed random variable that takes values in $\{3, 1, -1, -3\}$, $w(t)$ is the deterministic waveform

$$w(t) \quad = \quad \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 0, & \text{otherwise,} \end{cases}$$

and $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$.

(a) Describe the receiver that, based on the channel output $R(t)$, decides on the value of X with least probability of error.

(b) Find the probability of error of the receiver you have described in Part (a).

(c) Suppose now that you still use the receiver you have described in Part (a), but that the received signal is actually

$$R(t) \quad = \quad \frac{3}{4}\, c\, X\, w(t) + N(t),$$

i.e., you were unaware that the channel was attenuating the signal. What is the probability of error now?

(d) Suppose now that you still use the receiver you have found in Part (a) and that $R(t)$ is according to Equation (1), but that the noise is *colored*. In fact, $N(t)$ is a zero-mean stationary Gaussian noise process of auto-covariance function

$$K_N(\tau) = \frac{1}{4\alpha} e^{-|\tau|/\alpha},$$

where $0 < \alpha < \infty$ is some deterministic real parameter. What is the probability of error now?