Solution 1.

(a) In the absence of noise,

$$y(t) = \sum_{l \in \mathbb{Z}} s_l p(t - lT) \star h(t) \star q(t).$$

Because $f(t-\tau) \star g(t) = (f \star g)(t-\tau)$, we obtain

$$y(t) = \sum_{l \in \mathbb{Z}} s_l (p \star h \star q)(t - lT) = \sum_{l \in \mathbb{Z}} s_l g(t - lT).$$

(b) Following the hint, we obtain

$$\begin{aligned} g(-lT) &= \int_{-\infty}^{\infty} g_{\mathcal{F}}(f) e^{-\mathrm{j}2\pi f lT}\, df \\ &= \sum_{i \in \mathbb{Z}} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} g_{\mathcal{F}}(f - \frac{i}{T}) e^{-\mathrm{j}2\pi\left(f - \frac{i}{T}\right) lT}\, df \\ &= \sum_{i \in \mathbb{Z}} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} g_{\mathcal{F}}(f - \frac{i}{T}) e^{-\mathrm{j}2\pi f lT}\, df \\ &= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{i \in \mathbb{Z}} g_{\mathcal{F}}(f - \frac{i}{T}) e^{-\mathrm{j}2\pi f lT}\, df \\ &= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} s(f) e^{-\mathrm{j}2\pi f lT}\, df, \end{aligned}$$

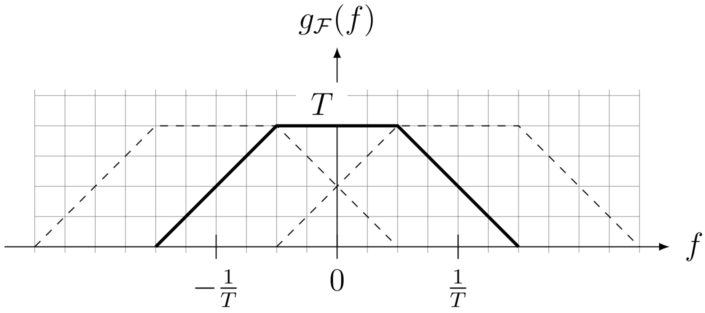
where $s(f) = \sum_{i \in \mathbb{Z}} g_{\mathcal{F}}(f - \frac{i}{T})$.

Therefore, $g(-lT)$ are the Fourier series coefficients (multiplied by $1/T$) of the periodic function $s(f)$. If $g(lT) = \mathbb{1}\{l = 0\}$, then $s(f)$ is constant over the interval $\left[-\frac{1}{2T}, \frac{1}{2T}\right]$ and equal to T. If $s(f)$ is constant and equal to T over $\left[-\frac{1}{2T}, \frac{1}{2T}\right]$, then $g(lT) = \mathbb{1}\{l = 0\}$.

Because $s(f)$ has period $1/T$, it is equal to T for any f, not only in the interval $\left[-\frac{1}{2T}, \frac{1}{2T}\right]$.

(c) Using the hint and (b), we see that $\{\psi(t - jT)\}_{j=-\infty}^{\infty}$ is an orthonormal set if and only if $\sum_{l \in \mathbb{Z}} R_{\psi\mathcal{F}}(f - \frac{l}{T}) = T$. But $R_{\psi\mathcal{F}}(f) = |\psi_{\mathcal{F}}(f)|^2$, which proves Theorem 5.6.

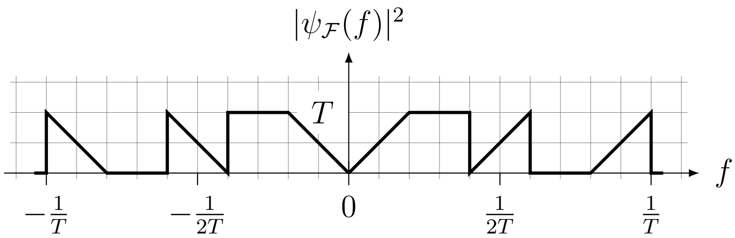
(d) From the below figure, we see that $\sum_{i\in\mathbb{Z}} g_{\mathcal{F}}(f-\frac{i}{T}) = 2T$. Therefore, according to (b), $g(kT) = 2\mathbb{1}\{k=0\}$ and $y(kT) = 2s_k$.



SOLUTION 2. Because $\psi(t)$ is real, its Fourier transform is conjugate symmetric $(\psi_{\mathcal{F}}(f) = \psi^*_{\mathcal{F}}(-f))$.

From the condition $\int \psi(t-kT)\psi(t-lT)dt = \mathbb{1}\{k=l\}$ for every pair k, l, it follows that $|\psi_{\mathcal{F}}(f)|^2$ satisfies Nyquist's criterion with parameter T, $\sum_{k\in\mathbb{Z}} |\psi_{\mathcal{F}}(f-k/T)|^2 = T$. On the other hand, since $\psi_{\mathcal{F}}(f) = 0$ for $|f| > \frac{1}{T}$, $|\psi_{\mathcal{F}}(f)|^2$ must have band-edge symmetry.

Putting everything together, we obtain the complete plot of $|\psi_{\mathcal{F}}(f)|^2$.



SOLUTION 3.

(a) The self-similarity function is the inverse Fourier transform of $|\psi_{\mathcal{F}}(f)|^2$, hence

$$|\psi_{\mathcal{F}}(f)|^2 = T\mathbb{1}\{f \in [-1/2T, 1/2T]\} \quad \Longleftrightarrow \quad R_{\psi}(\tau) = \operatorname{sinc}(\tau/T) = \frac{\sin(\pi\tau/T)}{\pi\tau/T}$$

To determine the time-domain pulse $\psi(t)$, we first observe that the frequency domain pulse can be

$$\psi_{\mathcal{F}}(f) = \sqrt{T}\mathbb{1}\{f \in [-1/2T, 1/2T]\}e^{-\mathrm{j}2\pi f t_0}, \qquad \text{for } \forall t_0 \in \mathbb{R}.$$

Consequently,

$$\psi(t) = \frac{1}{\sqrt{T}}\operatorname{sinc}\left(\frac{t-t_0}{T}\right) \qquad \text{for } \forall t_0 \in \mathbb{R}.$$

(b) We have

$$\begin{aligned}
Y_j &= \int_{-\infty}^{+\infty} R(t)\psi^*(t-jT)\ dt \\
&= \int_{-\infty}^{+\infty} W(t)\psi^*(t-jT) + \underbrace{\int_{-\infty}^{+\infty} N(t)\psi^*(t-jT)\ dt}_{\triangleq Z_j} \\
&= \int_{-\infty}^{+\infty} \sum_i S_i \psi(t-iT)\psi^*(t-jT)\ dt + Z_j \\
&= \sum_i S_i \int_{-\infty}^{+\infty} \psi(t-iT)\psi^*(t-jT)\ dt + Z_j \\
&= \sum_i S_i \int_{-\infty}^{+\infty} \psi(t-iT)\psi^*(t-jT)\ dt + Z_j \\
&= \sum_i S_i R_\psi\left((j-i)T\right) + Z_j. \qquad (1)
\end{aligned}$$

Since $N(t)$ is a white Gaussian noise we know Z_j are zero-mean jointly Gaussian random variables with covariance

$$\text{cov}(Z_i, Z_j) = \frac{N_0}{2}\int \psi^*(t-iT)\psi(t-jT) = \frac{N_0}{2} R_\psi\left((i-j)T\right). \qquad (2)$$

From Problem 1 we know that $\psi(t)$ is orthonormal to its time translates by T, which is equivalent to

$$R_\psi(kT) = \mathbb{1}\{k=0\}.$$

Using the above in (1) and (2) shows that

$$Y_j = S_j + Z_j,$$

where Z_j are i.i.d. $\mathcal{N}(0, \frac{N_0}{2})$ noise samples.

(c) Following the same steps as in (b) we have

$$\begin{aligned}
Y_j &= \int_{-\infty}^{+\infty} R(t)\psi^*(t - jT - \Delta)\ dt \\
&= \int_{-\infty}^{+\infty} W(t)\psi^*(t - jT - \Delta) + \underbrace{\int_{-\infty}^{+\infty} N(t)\psi^*(t - jT - \Delta)\ dt}_{\triangleq Z_j} \\
&= \int_{-\infty}^{+\infty} \sum_i S_i\psi(t - iT)\psi^*(t - jT - \Delta)\ dt + Z_j \\
&= \sum_i S_i \int_{-\infty}^{+\infty} \psi(t - iT)\psi^*(t - jT - \Delta)\ dt + Z_j \\
&= \sum_i S_i \int_{-\infty}^{+\infty} \psi(t - iT)\psi^*(t - jT - \Delta)\ dt + Z_j \\
&= \sum_i S_i R_\psi\left((j - i)T + \Delta\right) + Z_j. \qquad (3)
\end{aligned}$$

Once more since $N(t)$ is a white Gaussian noise we know Z_j are zero-mean jointly Gaussian random variables with covariance

$$\mathrm{cov}(Z_i, Z_j) = \frac{N_0}{2}\int \psi^*(t - iT - \Delta)\psi(t - jT - \Delta) = \frac{N_0}{2} R_\psi\left((i - j)T\right),$$

which shows Z_j are still i.i.d. $\mathcal{N}(0, \frac{N_0}{2})$ noise samples. However, the samples are now of the form

$$Y_j = l_0 S_j + \sum_{i \neq j} S_i l_{j-i} + Z_j$$

where

$$l_k = R_\psi(kT + \Delta) = \mathrm{sinc}\left(k + \frac{\Delta}{T}\right). \qquad (4)$$

(d) Suppose $\Delta = \frac{T}{2}$. Then,

$$l_k = \mathrm{sinc}\left(k + \frac{1}{2}\right) = \frac{\sin(k\pi + \pi/2)}{k\pi + \pi/2} = \frac{(-1)^k}{k\pi + \pi/2}.$$

Now if $s_1, \ldots, s_n$ is a sequence satisfying $s_i = -s_{i+1}$, for $i = 1, 2, \ldots, n-1$ and arbitrary $s_n \in \{\pm 1\}$, we have

$$\begin{aligned}
Y_n &= \frac{2}{\pi} s_n + \sum_{k \neq 0} s_{n-k} l_k + Z_n \\
&= \frac{2}{\pi} s_n + \sum_{k=1}^{n} s_{n-k} l_k + Z_n \\
&= \frac{2}{\pi} s_n + \sum_{k=1}^{n} (-1)^k \frac{(-1)^k}{k\pi + \pi/2} s_n + Z_n
\end{aligned}$$

The ISI term is hence

$$\sum_{k=1}^{n}(-1)^k \frac{(-1)^k}{k\pi+\pi/2} s_n = \sum_{k=1}^{n} \frac{1}{k\pi+\pi/2} s_n$$
$$= s_n \left(\sum_{k=0}^{n} \frac{1}{k\pi+\pi/2} - \frac{\pi}{2} \right)$$

We see that the term inside the parenthesis diverges as n grows large.

SOLUTION 4.

(a) When $i = j$, $\mathbb{E}[X_i X_j]$ is

$$\mathbb{E}\left[X_i^2\right] = \mathbb{E}[1] = 1.$$

Remember that the B_i are iid Bernoulli $(\frac{1}{2})$ random variables. Hence, we find immediately

$$\begin{aligned} K_X[1] &= \mathbb{E}[X_{2n}X_{2n+1}] = \mathbb{E}[B_n B_{n-2} B_n B_{n-1} B_{n-2}] \\ &= \mathbb{E}\left[B_n^2 B_{n-1} B_{n-2}^2\right] \\ &= \mathbb{E}[B_{n-1}] \;=\; 0, \end{aligned}$$

and also

$$\begin{aligned} K_X[2] &= \mathbb{E}[X_{2n}X_{2n+2}] = \mathbb{E}[B_n B_{n-2} B_{n+1} B_{n-1}] \\ &= \mathbb{E}[B_n]\,\mathbb{E}[B_{n-2}]\,\mathbb{E}[B_{n+1}]\,\mathbb{E}[B_{n-1}] \;=\; 0. \end{aligned}$$

By continuing this argument we find

$$K_X[i] = \mathbb{1}\{i = 0\}.$$

Hence,

$$S_X(f) = \frac{\mathcal{E}_s}{T_s} |\psi_{\mathcal{F}}(f)|^2.$$

This means that by choosing $\psi(t)$ appropriately, we can control the bandwidth consumption of our communications scheme.

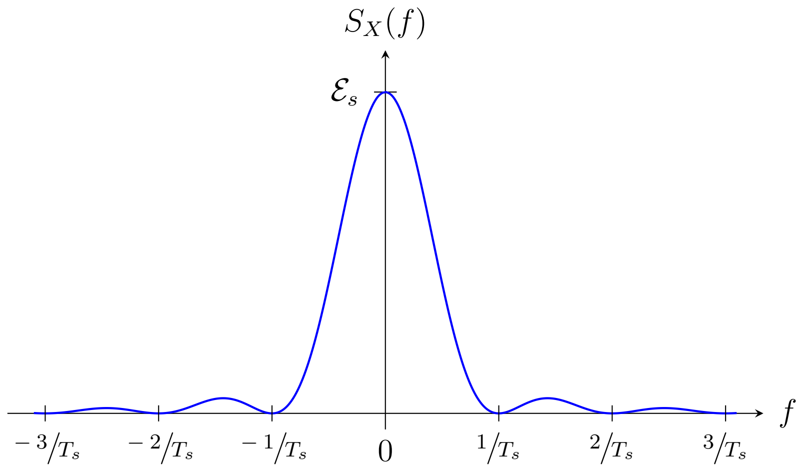
(b) We know that

$$|\psi_{\mathcal{F}}(f)|^2 = T_s \operatorname{sinc}^2(T_s f).$$

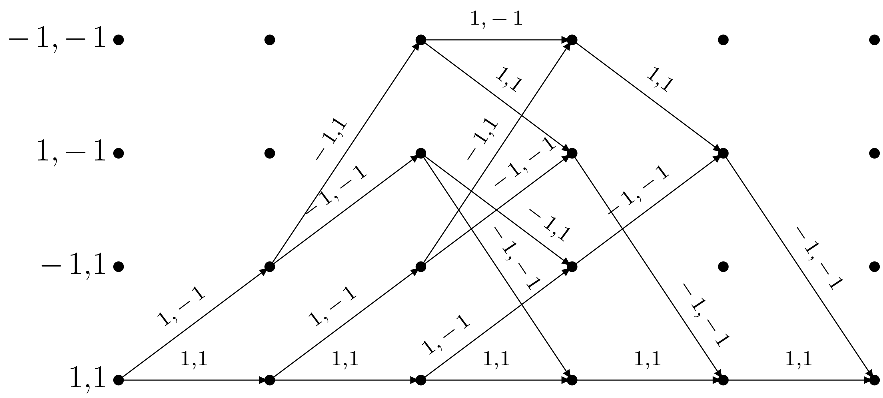
It follows that

$$S_X(f) = \mathcal{E}_s \operatorname{sinc}^2(T_s f).$$

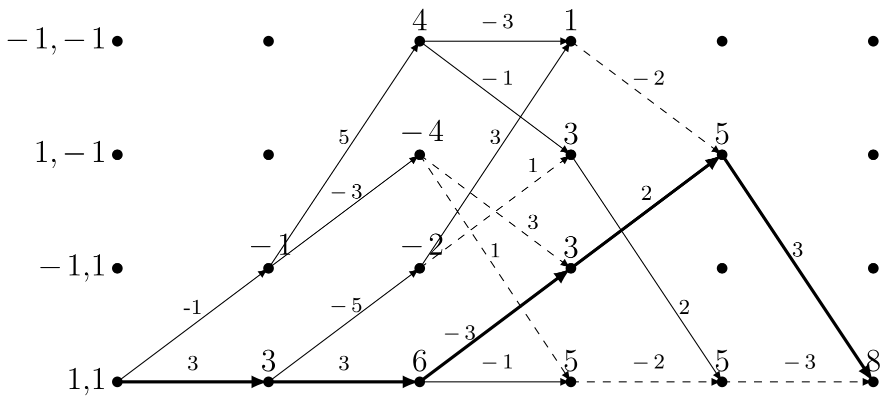
A plot of $S_X(f)$ is shown here:



SOLUTION 5. The trellis representing the encoder is shown below:



We display the diagram labeled with edge-metric according to the received sequence and state-metric of the survivor path. We also indicate the survivor paths and the decoding path.



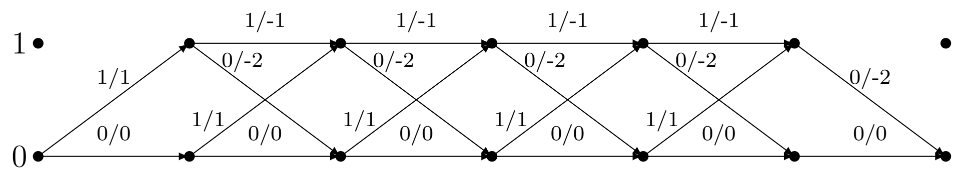
From the figure we can read the decoded sequence 1, 1, -1, 1, 1.

Solution 6.

(a)

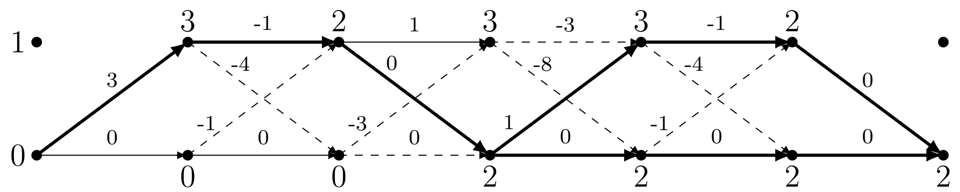
$$X_i = B_i - 2B_{i-1}$$

From this, we can draw the following trellis:



(b) We have $Y = X + Z$, where $Z = (Z_1, \ldots, Z_6)$ is a sequence of iid components with $Z_i \sim \mathcal{N}(0, \sigma^2)$. Our maximum likelihood decoder is a minimum distance decoder. We have to minimize $\|y - x\|^2$ or equivalently, maximize $2\langle y, x\rangle - \|x\|^2$. We thus have $f(x, y) = \sum_{i=1}^{6} 2y_i x_i - x_i^2$ whose maximization with respect to x leads to a maximum likelihood decision on X.

(c) We label our trellis with the edge metric $2y_i x_i - x_i^2$ and then trace back the decoding path.



We see that the two sequences $1, 1, 0, 0, 0$ and $1, 1, 0, 1, 1$ are equally likely, so the decoder would choose either of the two.