ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 2 | Principles of Digital Communications
Problem Set 1 | Feb. 18, 2015

Problem Set 1

PROBLEM 1. *(Probabilities of Basic Events)* Assume that X_1 and X_2 are independent random variables that are uniformly distributed in the interval $[0,1]$. Compute the probability of the following events. Hint: For each event, identify the corresponding region inside the unit square.

(a) $0 \leq X_1 - X_2 \leq \frac{1}{3}$.

(b) $X_1^3 \leq X_2 \leq X_1^2$.

(c) $X_2 - X_1 = \frac{1}{2}$.

(d) $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq (\frac{1}{2})^2$.

(e) Given that $X_1 \geq \frac{1}{4}$, compute the probability that $(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \leq (\frac{1}{2})^2$.

PROBLEM 2. *(Basic Probabilities)* Find the following probabilities.

(a) A box contains m white and n black balls. Suppose k balls are drawn. Find the probability of drawing at least one white ball.

(b) We have two coins; the first is fair and the second is two-headed. We pick one of the coins at random, we toss it twice and heads shows both times. Find the probability that the coin is fair.

PROBLEM 3. *(Conditional Distribution)* Assume that X and Y are random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} A, & 0 \leq x < y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent?

(b) Find the value of A.

(c) Find the density function of Y. Do this first by arguing geometrically then compute it formally.

(d) Find $\mathbb{E}[X|Y=y]$. Hint: Argue geometrically.

(e) The $\mathbb{E}[X|Y=y]$ found in Part (d) is a function of y, call it $f(y)$. Find $\mathbb{E}[f(Y)]$. This is $\mathbb{E}[\mathbb{E}[X|Y]]$.

(f) Find $\mathbb{E}[X]$ from the definition. Verify that $\mathbb{E}[X]$ is equal to $\mathbb{E}[\mathbb{E}[X|Y]]$ that you have found in Part (e). Is this a coincidence?

PROBLEM 4. *(Gaussian Random Variables)* Let Z_1 and Z_2 be i.i.d. zero-mean Gaussian random variables, i.e. the pdf of Z_i, $i=1,2$ is

$$f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}.$$

for some $\sigma > 0$. Define

$$X := \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2}} \qquad \text{and} \qquad Y := \frac{Z_2}{\sqrt{Z_1^2 + Z_2^2}}.$$

Prove that (X,Y) is a uniformly chosen point on the unit circle.

PROBLEM 5. *(Uncorrelated vs. Independent Random Variables)*

(a) Let X and Y be two continuous real-valued random variables with joint probability density function f_{XY}. Show that if X and Y are independent, they are also uncorrelated.

(b) Consider two independent and uniformly distributed random variables $U \in \{0,1\}$ and $V \in \{0,1\}$. Assume that X and Y are defined as follows: $X = U + V$ and $Y = |U - V|$. Are X and Y independent? Compute the covariance of X and Y. What do you conclude?

PROBLEM 6. *(Properties of Expectation)* Assume you pick a point on the surface of the unit sphere (i.e. the sphere centered at the origin with radius 1) uniformly at random and (X,Y,Z) denotes its coordinates (in 3D space). Compute $\mathbb{E}[X^2]$.