on to Chapter 8 / 7:
a goody bag of assorted fancy counting tricks, without proofs

Counting using recurrence relations

pages 485-494 /449-456

section 8.1 / 7.1:

examples of non-obvious counting problems that allow easy reduction to sub-problems

general approach:

- solution a_n to problem of size n is written as function f of $a_1, a_2, \ldots, a_{n-1}$
- depending on f this may (or may not) lead to a way to determine a_n (in later sections)

examples: runtimes from earlier sections

- binary search among n items in $b_n = b_{n/2} + C$
- mergesort of n items in $m_n = 2m_{n/2} + n$,

solved using ad hoc techniques and MI

Counting examples, section 8.1 / 7.1

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compound interest:

- deposit $d_0 = x$ at 2% interest, d_n after n years: clearly $d_n = 1.02 d_{n-1}$ and thus $d_n = 1.02^n x$
- with additional annual contribution of y: $d_n = 1.02 d_{n-1} + y$, general solution more work

(undying) rabbits, or # n-bitstrings without "00":

$a_n = a_{n-1} + a_{n-2}$ (Fibonacci, different a_1, a_2)

towers of Hanoi: $h_1 = 1$, $h_n = 2h_{n-1} + 1 = \ldots = 2^n - 1$

n-digit integers with even number of zeros:

$$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$$

parenthizations of $x_0 * x_1 * \ldots * x_n$ (Catalan):

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0$$

8.2 / 7.2: Solving (some of) these recurrences

pages 497-508 /460-471

solving $a_n = c_1 a_{n-1}$ was easy: $a_n = (c_1)^n a_0$

next case, $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, is harder:
is degree 2 case of

linear homogeneous recurrence relation of degree *k*:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where c_is are real constants and $c_k \neq 0$

- ✓ $d_n = 1.02 d_{n-1}$, of degree 1
- • $d_n = 1.02 d_{n-1} + y$: nonhomogeneous
- ✓ $a_n = a_{n-1} + a_{n-2}$, of degree 2
- • $h_n = 2h_{n-1} + 1$: nonhomogeneous
- • $m_n = 2m_{n/2} + n$: no fixed degree, nonhomogeneous
- • $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-1} C_0$: nonlinear

Solving $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ ($c_2 \neq 0$)

pages 498-501 /461-464

try $a_n = r^n$ as solution (for unknown $r \neq 0$):

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \Leftrightarrow r^n = c_1 r^{n-1} + c_2 r^{n-2}$$

$$\Leftrightarrow r^n - c_1 r^{n-1} - c_2 r^{n-2} = 0$$

$$\Leftrightarrow r^2 - c_1 r - c_2 = 0$$

$\Rightarrow \forall\, r$: $r^2 - c_1 r - c_2 = 0$ and $\alpha_r \in \mathbf{R}$:

$$a_n = \Sigma_r\, \alpha_r r^n \text{ solves } a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

polynomial $r^2 - c_1 r - c_2$ has 2 or 1 roots:

- 2 roots: r_1, r_2 with $r_1 \neq r_2$, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
- single root r: $a_n = \alpha_1 r^n + \alpha_2 n r^n$ (root of derivative as well)

with α_i determined by a_0 and a_1

conversely: each solution of this form

Example: solving $f_n = f_{n-1} + f_{n-2}$, $f_0=0, f_1=1$

pages 500-501 /463-464

$r^2 - r - 1$ has roots $(1\pm\sqrt{5})/2$

$\Rightarrow f_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$

from $f_0 = \alpha_1 + \alpha_2 = 0$
and $f_1 = \alpha_1(1+\sqrt{5})/2 + \alpha_2(1-\sqrt{5})/2 = 1$
it follows that $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$

$\Rightarrow$ the nth Fibonacci number is
$f_n = (\,(1+\sqrt{5})^n - (1-\sqrt{5})^n\,)/(2^n\sqrt{5})$

Example: solving $d_n = 4d_{n-1} - 4d_{n-2}$, $d_0 = d_1 = 1$

pages 500-501 /463-464

$r^2 - 4r + 4 = (r-2)^2$ has double root 2

$\Rightarrow d_n = \alpha_1 2^n + \alpha_2 n 2^n$

from $d_0 = \alpha_1 = 1$

and $d_1 = \alpha_1 2 + \alpha_2 2 = 1$

it follows that $\alpha_1 = 1$ and $\alpha_2 = -½$

$\Rightarrow d_n = 2^n - n2^{n-1}$

pages 502-503 /465-466

Solving $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ ($c_k \neq 0$)

same approach: try $a_n = r^n$ as solution …:

polynomial $r^k - c_1 r^{k-1} - \ldots - c_k$ has $\leq k$ roots:

- all distinct roots $r_1, r_2, \ldots, r_k$:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

- roots with multiplicities: more complicated

with α_i determined by $a_0, a_1, \ldots, a_{k-1}$

conversely: each solution of this form

Note:

- $r^k - c_1 r^{k-1} - \ldots - c_k$: **characteristic equation**
- its roots the **characteristic roots**

Handling the nonhomogeneous case

pages 504-507 /467-469

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$

with c_is real constants, $c_k \neq 0$, and $F(n) \neq 0$:

linear nonhomogeneous recurrence relation

with $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ as its

associated homogeneous recurrence relation

- any solution to the latter (which we know) can be added to solution $a_n^{(p)}$ to the former
- **particular solution** $a_n^{(p)}$ always exists if

  $F(n) = (\text{degree } t \text{ polynomial in } n) \times s^n$:

  $$a_n^{(p)} = n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_0) s^n$$

  where $p_t \neq 0$ and m is multiplicity of s as root of $r^k - c_1 r^{k-1} - \ldots - c_k$

Example application "nonhomogeneous"

pages 504-507 /467-469

Hanoi, h_1=1, h_n =$2h_{n-1}$+1= …= 2^n–1:

- characteristic equation ("CE") $r - 2 = 0$

$\Rightarrow$ solution to homogeneous part is $\alpha 2^n$

- $F(n) = 1$ = (degree t polynomial in n) × s^n

  $t = 0$, $s = 1$:

  $\Rightarrow a_n^{(p)} = n^m (p_0) 1^n$ a particular solution

  $s = 1$ is not a root of CE, so $m = 0$

  substitute $a_n^{(p)} = p_0$ in $h_n = 2h_{n-1}+1$

  $\Rightarrow p_0 = -1$

- general solution of form $\alpha 2^n - 1$

  use h_1=1 $\Rightarrow \alpha 2^1 - 1 = 1 \Rightarrow \alpha = 1$

- solution is $2^n - 1$

Another application of "nonhomogeneous"

pages 504-507 /467-469

interest, $d_0 = x$, $d_n = 1.02d_{n-1} + y$, where $y \neq 0$:

- characteristic equation ("CE") $r - 1.02 = 0$

$\Rightarrow$ solution to homogeneous part is $\alpha 1.02^n$

- $F(n) = y$ = (degree t polynomial in n) $\times s^n$

  $t = 0$, $s = 1$:

  $\Rightarrow a_n^{(p)} = n^m (p_0) 1^n$ a particular solution

  $s = 1$ is not a root of CE, so $m = 0$

  substitute $a_n^{(p)} = p_0$ in $d_n = 1.02d_{n-1} + y$

  $\Rightarrow p_0 = -y/0.02$

- general solution of form $\alpha 1.02^n - y/0.02$

  use $d_0 = x \Rightarrow \alpha 1.02^0 - y/0.02 = x \Rightarrow \alpha = x + y/0.02$

- solution is $(x + y/0.02)1.02^n - y/0.02$

Another example

pages 504-507 /467-469

a_n: the number of length n ternary strings (0s, 1s, 2s) with an even number of 1s

$n = 0$: “” is unique empty string, no 1s: $a_0 = 1$

$n = 1$: “0” and “2”, thus $a_1 = 2$

$n = 2$: “00”, “02”, “11”, “20”, “22”, thus $a_2 = 5$

recurrence relation?

to get a proper length $n+1$ string:

- take any (of 3^n) ternary length n string, add a “0” (if even 1s) or “1” (if odd 1s)
- or add “2” to any of the a_n length n strings

$\Rightarrow a_{n+1} = a_n + 3^n$, example of nonhomogeneous

(& confirming that $a_0 = 1$)

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Solving nonhomogeneous $a_{n+1} = a_n + 3^n$

(with $a_0 = 1$, $a_1 = 2$, $a_2 = 5$)

characteristic equation ("CE") $r - 1 = 0$

$\Rightarrow$ Solution to homogeneous part is $\alpha 1^n = \alpha$

- $F(n) = 3^n =$ (degree t polynomial in n) $\times s^n$

  $t = 0$, $s = 3$:

  $\Rightarrow a_n^{(p)} = n^m (p_0) 3^n$ a particular solution

  $s = 3$ is not a root of CE, so $m = 0$

  substitute $a_n^{(p)} = p_0 3^n$ in $a_{n+1} = a_n + 3^n$

  $\Rightarrow p_0 3^{n+1} = p_0 3^n + 3^n \Rightarrow p_0 = ½$

- general solution of form $\alpha + ½\, 3^n$

  use $a_0 = 1 \Rightarrow \alpha + ½ = 1 \Rightarrow \alpha = ½$

- solution is $½(1 + 3^n)$ (is intuitively about right)

Final section 8.2 / 7.2 example application

pages 507 /470-471

sum of squares $a_n = \sum_{i=1}^{n} i^2 \Rightarrow a_n = a_{n-1} + n^2$

- characteristic equation ("CE") $r - 1 = 0$

$\Rightarrow$ solution to homogeneous part is $\alpha 1^n = \alpha$

- $F(n) = n^2 =$ (degree 2 polynomial in n) $\times s^n$
  $t = 2, s = 1 \Rightarrow a_n^{(p)} = n^m (p_2 n^2 + p_1 n + p_0) 1^n$
  is a particular solution; since $s = 1$ is a root of CE of multiplicity 1 it follows that $m = 1$
- substitute $a_n^{(p)} = n(p_2 n^2 + p_1 n + p_0)$
  in $a_n = a_{n-1} + n^2$
  and solve for p_0, p_1, p_2 using a_0, a_1, a_2
  (very much like we've determined a_n before)
- finally use solution $a_n^{(p)} + \alpha$ to conclude α=0

Brief remark on 8.3 / 7.3: simple tricks

pages 511-519 /474-482

so far: linear (non)homogeneous recurrences
of fixed degree

not suitable to solve

- $b_n = b_{n/2} + C$ (binary search runtime)
  $b_n = O(\log n)$
- $m_n = 2m_{n/2} + n$ (mergesort runtime)
  $m_n = O(n \log n)$
- $k_n = 3k_{n/2} + Cn$ (Karatsuba runtime)
  $k_n = O\left(n^{\log_2 3}\right)$
- $s_n = 7s_{n/2} + Cn^2$ (Strassen's Karatsuba-like matrix ×)
  $s_n = O\left(n^{\log_2 7}\right)$
- etc: see: thms 1&2, pages 514&516 / 477&479
  (or use common sense)

8.4 / 7.4 Generating functions

pages 520-532 /484-495

solving counting problems
by interpreting coefficients of polynomials (or power series) as the required solutions

simple example:
number of non-negative integer solutions to

$$e_1 + e_2 + e_3 = 4$$

- pick 4 cookies from 3 types of cookies in 3+4–1 choose 3–1 = 15 ways
- pick x^{e_1} from $1+x+x^2+x^3+\ldots = 1/(1-x)$,
  pick x^{e_2} from $1+x+x^2+x^3+\ldots = 1/(1-x)$, and
  pick x^{e_3} from $1+x+x^2+x^3+\ldots = 1/(1-x)$

$\Rightarrow$ need coefficient of x^4 in $1/(1-x)^3$

Power series, basics

pages
520-532
/484-495

- power series is a polynomial of possibly infinite degree:

$$f(x) = \sum_{i=0}^{\infty} f_i x^i, \; g(x) = \sum_{j=0}^{\infty} g_j x^j$$

- define: $f(x) + g(x) = \sum_{i=0}^{\infty} (f_i + g_i) x^i$

$$f(x)g(x) = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} f_k g_{i-k} \right) x^i$$

...

- 1-to-1 correspondence between h and its **Taylor series expansion** (around a):

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!} (x-a)^n \; (h^{(n)} : n\text{th derivative})$$

Common power/Taylor series expansions

page 526/489

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k \qquad \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$e^x = \sum_{k=0}^{\infty} x^k / k! \qquad \ln(1+x) = \sum_{k=0}^{\infty} (-1)^{k+1} x^k / k$$

plus common substitutions ($-x$ or cx for x), and term by term differentiation and integration

Back to simple example

pages 520-532 /484-495

number of non-negative integer solutions to

$$e_1 + e_2 + e_3 = 4$$

- pick 4 cookies from 3 types of cookies in 3+4–1 choose 3–1 = 15 ways
- pick x^{e_1} from $1 + x + x^2 + x^3 + \ldots = 1/(1-x)$, pick x^{e_2} from $1 + x + x^2 + x^3 + \ldots = 1/(1-x)$, and pick x^{e_3} from $1 + x + x^2 + x^3 + \ldots = 1/(1-x)$

$\Rightarrow$need coefficient of x^4 in $1/(1-x)^3$

with $$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

this coefficient equals $\binom{3+4-1}{4} = 15$

Another simple example

pages
520-532
/484-495

number of non-negative integer solutions to

$$e_1 + e_2 + e_3 = 4$$

such that e_2 is even and e_3 a multiple of 3

- unclear how to use basic earlier method
- pick x^{e_1} from $1 + x + x^2 + x^3 + \ldots = 1/(1-x)$,
  pick x^{e_2} from $1 + x^2 + x^4 + \ldots = 1/(1-x^2)$, and
  pick x^{e_3} from $1 + x^3 + x^6 + \ldots = 1/(1-x^3)$

$\Rightarrow$ need coefficient of x^4 in $1/((1-x)(1-x^2)(1-x^3))$

Final simple example

pages
520-532
/484-495

number of non-negative integer solutions to

$$e_1 + e_2 + e_3 + e_4 = 20$$

with e_1 even, e_2 multiple of 5, $e_3 \le 4$, $e_4 \le 1$,

x^{e_1} from $1 + x^2 + x^4 + ... = 1/(1-x^2)$, and

x^{e_2} from $1 + x^5 + x^{10} + ... = 1/(1-x^5)$

x^{e_3} from $1 + x + x^2 + x^3 + x^4 = (1-x^5)/(1-x)$

x^{e_4} from $1 + x = (1-x^2)/(1-x)$

$\Rightarrow$ solution is 21 : the coefficient of x^{20} in

$$\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)\left(\frac{1-x^2}{1-x}\right) = \frac{1}{(1-x)^2}$$

Generating function to solve recurrences

page 530/493

let $a_0 = 1$, $a_n = 2a_{n-1}$

show that $a_n = 2^n$ using a generating function:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i \Rightarrow xA(x) = \sum_{j=1}^{\infty} a_{j-1} x^j$$

$$\Rightarrow A(x) - 2xA(x) = a_0 + \sum_{i=1}^{\infty} (a_i - 2a_{i-1}) x^i = 1$$

$$\Rightarrow A(x) = 1/(1-2x)$$

we know that $\sum_{i=0}^{\infty} r^i = 1/(1-r)$ (for $|r| \leq 1$)

with $r = 2x$ we find $\sum_{i=0}^{\infty} (2x)^i = 1/(1-2x)$

and thus $\sum_{i=0}^{\infty} (2x)^i = A(x)$

it follows that $a_i = 2^i$

Another example: $a_n = a_{n-1} + n,\ a_0 = 0, a_1 = 1$

page
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$$A(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + \sum_{i=1}^{\infty} a_i x^i = \sum_{i=1}^{\infty} a_i x^i$$

$$\Rightarrow A(x) = \sum_{i=1}^{\infty} (a_{i-1} + i) x^i = \sum_{j=0}^{\infty} (a_j + j + 1) x^{j+1}$$

$$\Rightarrow A(x) = \sum_{j=0}^{\infty} a_j x^{j+1} + \sum_{j=0}^{\infty} (j+1) x^{j+1}$$

$$\Rightarrow A(x) = x\sum_{j=0}^{\infty} a_j x^j + x\sum_{j=0}^{\infty} (j+1) x^j$$

$$\Rightarrow A(x) = xA(x) + \frac{x}{(1-x)^2}$$

(← and ↓ use page 526/489)

$$\Rightarrow A(x) = \frac{x}{(1-x)^3} = \sum_{i=0}^{\infty} C(3+i-1, i) x^{i+1}$$

$$\Rightarrow A(x) = \sum_{j=1}^{\infty} C(j+1, j-1) x^j \Rightarrow a_n = \frac{n(n+1)}{2}$$

The approach:

pages 530-532 /493-495

- interpret sequence a_n to be determined as coefficients of a power series of some A
- use the recurrence relation to derive an alternative expression f for A
- find (using a table, using Taylor, …) power series expansion for f:

$$f(x) = \sum_{i=0}^{\infty} f_i x^i$$

- coefficients f_i are closed expression for a_i

(many more details in section 8.4 / 7.4)

$a_{n+1} = a_n + 3^n$ with generating functions

page 530/493

$$a_0 = 1,\ a_1 = 2,\ a_2 = 5$$

$$A(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + \sum_{i=1}^{\infty} a_i x^i$$

$$\Rightarrow A(x) = 1 + \sum_{j=0}^{\infty} a_{j+1} x^{j+1} = 1 + x \sum_{j=0}^{\infty} a_{j+1} x^j$$

$$\Rightarrow A(x) = 1 + x \sum_{j=0}^{\infty} a_j x^j + x \sum_{j=0}^{\infty} 3^j x^j$$

$$\Rightarrow A(x) = 1 + xA(x) + \frac{x}{1-3x}$$

$$\Rightarrow (1-x)A(x) = 1 + \frac{x}{1-3x} = \frac{1-2x}{1-3x}$$

$$\Rightarrow A(x) = \frac{1-2x}{(1-x)(1-3x)}$$

Continuation

page 530/493

we have $A(x) = \dfrac{1-2x}{(1-x)(1-3x)}$

write $A(x) = \dfrac{u}{1-x} + \dfrac{v}{1-3x}$

thus $u(1-3x) + v(1-x) = 1-2x$

implying that $u+v=1$ and $3u+v=2$

thus $u=v=1/2$ and $A(x) = \frac{1}{2}\left(\dfrac{1}{1-x} + \dfrac{1}{1-3x}\right)$

it follows that $A(x) = \sum_{i=0}^{\infty} \frac{1}{2}(1^i + 3^i)x^i$

and thus that $a_n = \frac{1}{2}(1+3^n)$

(always check correctness of a_0, a_1, a_2)

Catalan numbers

page ?/498

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} \text{ with } C_0 = C_1 = 1$$

$$\text{let } G(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$\Rightarrow G(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} C_k C_{n-1-k} \right) x^{n-1}$$

$$= \sum_{n=1}^{\infty} C_n x^{n-1}$$

$$\Rightarrow xG(x)^2 = \sum_{n=1}^{\infty} C_n x^n = G(x) - C_0$$

$$\Rightarrow xG(x)^2 - G(x) + 1 = 0$$

$$\Rightarrow G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Catalan numbers, continued

$$G(x)=\frac{1-\sqrt{1-4x}}{2x}$$ (+ - choice is bad at zero)

let $xG(x)=\frac{1}{2}-\frac{1}{2}\sqrt{1-4x}=f(x)$

$$\Rightarrow f'(x)=(1-4x)^{-1/2}$$

we will see that $(1-4x)^{-1/2}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n$

$\Rightarrow f'(x)=\sum_{n=0}^{\infty}\binom{2n}{n}x^n$, term by term integration :

$$\Rightarrow f(x)=c+\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}$$

$$\Rightarrow c+\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}=xG(x)=\sum_{n=0}^{\infty}C_n x^{n+1}$$

$$\Rightarrow C_n=\frac{1}{n+1}\binom{2n}{n}$$

Extended binomial coefficients and theorem

n,k integers ≥ 0: $\binom{n}{k} = \dfrac{n(n-1)\ldots(n-k+1)}{k!}$

define for real u and integer $k > 0$:

$$\binom{u}{k} = \frac{u(u-1)\ldots(u-k+1)}{k!} \text{ and } \binom{u}{0} = 1$$

then for any real u and real x with $|x| < 1$:

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

compare to binomial theorem (integer $n \geq 0$):

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Back to Catalan numbers, $(1–4x)^{-½}$

from $(1+x)^u = \sum_{n=0}^{\infty} \binom{u}{n} x^n$ it follows that

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-4x)^n$$

we will see that $\binom{-1/2}{n} = \binom{2n}{n} \frac{1}{(-4)^n}$

thus $(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(-4)^n} (-4x)^n$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

Final step: $\binom{-1/2}{n}$

for positive integer n:

$$\binom{-1/2}{n} = \frac{(-1/2)((-1/2)-1)\ldots((-1/2)-n+1)}{n!}$$

$$= \frac{(-1/2)(-3/2)(-5/2)\ldots(-(2n-1)/2)}{n!}$$

$$= (-1)^n \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2^n n!}\cdot\frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{2^n n!}$$

$$= (-1)^n \frac{1\cdot 2\cdot 3\cdot \ldots \cdot (2n)}{4^n n! n!}$$

$$\Rightarrow \binom{-1/2}{n} = \binom{2n}{n}\frac{1}{(-4)^n}$$

8.5 & 8.6 / 7.5 & 7.6: Inclusion & Exclusion

pages 535-547 /499-512

covered in homeworks and at midterm