Handout 8 | Information Theory and Coding
Homework 4 (Graded - Due on Oct. 20, 2014) | Oct. 07, 2014

Problem 1. (20 pts)

(a) (10 pts) Let U be a random variable taking values in the alphabet $\mathcal{U}$, and let f be a mapping from $\mathcal{U}$ to $\mathcal{V}$. Show that $H\big(f(U)\big) \le H(U)$.

(b) (10 pts) Let U and V be two random variables taking values in the alphabets $\mathcal{U}$ and $\mathcal{V}$ respectively, and let f be a mapping from $\mathcal{V}$ to $\mathcal{W}$. Show that $H(U|V) \le H\big(U\big|f(V)\big)$.

Problem 2. (15 pts)

(a) (10 pts) Let U and $\hat{U}$ be two random variables taking values in the same alphabet $\mathcal{U}$, and let $p_e = \mathbb{P}[U \ne \hat{U}]$. Show that $H(U|\hat{U}) \le h(p_e) + p_e \log(|\mathcal{U}| - 1)$, where $h(p) = p \log \frac{1}{p} + (1-p)\log\frac{1}{1-p}$. Hint: use the random variable $W \in \{0,1\}$ defined by

$$W = \begin{cases} 1 \text{ if } U \ne \hat{U}, \\ 0 \text{ otherwise.} \end{cases}$$

(b) (5 pts) Let U and V be two random variables taking values in the alphabets $\mathcal{U}$ and $\mathcal{V}$ respectively, and let f be a mapping from $\mathcal{V}$ to $\mathcal{U}$. Define $p_e = \mathbb{P}[U \ne f(V)]$. Show that $H(U|V) \le h(p_e) + p_e \log(|\mathcal{U}| - 1)$.

Problem 3. (20 pts) Let U and W be two independent random variables uniformly distributed in $\{0,1\}$, and $W' = U \oplus W$. Moreover, Z and Z' are two independent Bernoulli(p) random variables in $\{0,1\}$, where $p \in \big(0, \frac{1}{2}\big)$. We also assume that Z and Z' are independent from U and W. Define $V = U \oplus Z$ and $V' = U \oplus Z'$.

(a) (5 pts) Show that $U \multimap V \multimap W$ and $U \multimap V' \multimap W'$ are Markov chains. Deduce that $I(U;V) \ge I(U;W)$ and $I(U;V') \ge I(U;W')$.

(b) (5 pts) Compute $I(U;V)$, $I(U;W)$, $I(U;V')$ and $I(U;W')$. Deduce that $I(U;V) > I(U;W)$ and $I(U;V') > I(U;W')$.

(c) (10 pts) Compute $I(U;VV')$ and $I(U;WW')$. Deduce that $I(U;VV') < I(U;WW')$.

Problem 4. (15 pts)

(a) (10 pts) Show that for every $p, q \ge 0$, we have $\frac{1}{2}\left(p \log \frac{1}{p} + q \log \frac{1}{q}\right) \le \frac{p+q}{2} \log \frac{2}{p+q}$.

(b) (5 pts) The entropy $H(U)$ of a random variable U is a function of the distribution p_U of the random variable. Denote by $h(p)$ the entropy of a random variable with distribution p, i.e., $h(p) = \sum_{u \in \mathcal{U}} p(u) \log \frac{1}{p(u)}$. Let p and q be two probability distributions on the same alphabet $\mathcal{U}$, and let r be the probability distribution on $\mathcal{U}$ defined by $r(u) = \frac{p(u)+q(u)}{2}$ for every $u \in \mathcal{U}$. Show that $H(r) \ge \frac{1}{2}H(p) + \frac{1}{2}H(q)$.

PROBLEM 5. (30 pts) Consider a source U with alphabet $\mathcal{U}$ and suppose that we know that the true distribution of U is either P_1 or P_2. Define $S = \sum_{u \in \mathcal{U}} \max\{P_1(u), P_2(u)\}$.

(a) (10 pts) Show that $S \leq 2$ and give a necessary and sufficient condition for equality.

(b) (5 pts) Show that there exists a prefix-free code where the length of the codeword associated to each symbol $u \in \mathcal{U}$ is $l(u) = \left\lceil \log_2 \frac{S}{\max\{P_1(u), P_2(u)\}} \right\rceil$.

(c) (5 pts) Show that the average length $\bar{l}$ (using the true distribution) of the code constructed in (b) satisfies $H(U) \leq \bar{l} < H(U) + \log S + 1 \leq H(U) + 2$.

Now assume that the true distribution of U is one of k distributions $P_1, \dots, P_k$.

(d) (10 pts) Show that there exists a prefix-free code satisfying $H(U) \leq \bar{l} < H(U) + \log_2 S + 1 \leq H(U) + \log_2 k + 1$, where $S = \sum_{u \in \mathcal{U}} \max\{P_1(u), \dots, P_k(u)\}$.