

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6
Homework 3

Information Theory and Coding
Sep. 30, 2013

PROBLEM 1. Let $p_{XY}(x, y)$ be given by

$X \backslash Y$	0	1
0	1/3	1/3
1	0	1/3

Find

- (a) $H(X), H(Y)$.
- (b) $H(X|Y), H(Y|X)$.
- (c) $H(X, Y)$.
- (d) $H(Y) - H(Y|X)$.
- (e) $I(X; Y)$.
- (f) Draw a Venn diagram for the quantities in (a) through (e).

PROBLEM 2. Let X be a random variable taking values in M points a_1, \dots, a_M , and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha)H(Y)$$

where Y is a random variable taking values in $M - 1$ points a_1, \dots, a_{M-1} with probabilities $P_Y(a_j) = P_X(a_j)/(1 - \alpha)$; $1 \leq j \leq M - 1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)$$

and determine the condition for equality.

PROBLEM 3. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

- (a) $I(X, Y; Z) \geq I(X; Z)$.
- (b) $H(X, Y|Z) \geq H(X|Z)$.
- (c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.
- (d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

PROBLEM 4. For a stationary process X_1, X_2, \dots , show that

$$(a) \quad \frac{1}{n}H(X_1, \dots, X_n) \leq \frac{1}{n-1}H(X_1, \dots, X_{n-1}).$$

$$(b) \quad \frac{1}{n}H(X_1, \dots, X_n) \geq H(X_n|X_{n-1}, \dots, X_1).$$

PROBLEM 5. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0|X_{-1}, \dots, X_{-n}) = H(X_0|X_1, \dots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 6. Show, for a Markov chain, that

$$H(X_0|X_n) \geq H(X_0|X_{n-1}), \quad n \geq 1.$$

Thus, initial state X_0 becomes more difficult to recover as time goes by.