Graded Problem Set 11

Date: 28.11.2014 Due date: 05.12.2014

Instructions: Please write the solution for each problem on a separate piece of paper. Sort these pieces of paper, with Problem 1 on top. Add this page as cover page, fill in your name, Sciper number, and the list of collaborators, and staple them all together on the left top.

Rules: You are allowed and encouraged to discuss these problems with your colleagues. However, each of you has to write down her solution in her own words. If you collaborated on a homework, write down the name of your collaborators and your sources in the space below. No points will be deducted for collaborations. But if we find similarities in solutions beyond the listed collaborations we will consider it as cheating. Please note that EPFL has a VERY strict policy on cheating and you might be in BIG trouble. It is simply not worth it.

Grading: You will receive 10 bonus points additionally if you solve Problem 6.

Collaborators and sources:

First name: ....

Last name: ....

Sciper: ....

Problem 1	... / 20
Problem 2	... / 15
Problem 3	... / 15
Problem 4	... / 25
Problem 5	... / 25
Problem 6	... / 10
TOTAL	... / 100

Problem 1. Find the number of subsets of $S = \{1, 2, 3, \ldots, 10\}$ that

(a) contain neither 5 nor 6.

(b) contain no odd numbers.

(c) contain exactly five elements all of them even.

(d) contain exactly five elements including 3 or 4 but not both.

(e) contain exactly four elements, the sum of which is even.

Problem 2. Consider the equation

$$x + y + z = 32.$$

Find the number of solutions to this equation if

(a) x, y, and z are *positive* integers.

(b) x, y, and z are *non-negative* integers.

(c) x, y, and z are *non-negative* integers, $x \geq 7$ and $y \geq 15$.

Problem 3. Prove that for any prime number p and integers a and b,

$$(a + b)^p \equiv a^p + b^p \pmod p$$

Problem 4. Solve the following recurrence relations.

(a) $a_n = 15a_{n-1} - 56a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

(b) $a_n = 2a_{n-1} + 15a_{n-2}$ with $a_0 = a_1 = 2$.

(c) $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = a_1 = 1$.

(d) $a_n = 7a_{n-1} - 14a_{n-2} + 8a_{n-3}$ with $a_0 = 3/4$, $a_1 = 1$, and $a_2 = 3$.

(e) $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 0$, $a_1 = -1$, and $a_2 = 11$.

Problem 5. Consider a sequence whose generating function is $F(x)$ as below. Find a Θ-approximation for this sequence.

(a) $F(x) = \dfrac{x}{1 - 15x + 56x^2}$.

(b) $F(x) = \dfrac{x^{32}}{1 - 15x + 56x^2}$.

(c) $F(x) = \dfrac{1}{(x^2 - 9x + 14)(x^2 + 2x - 3)}$.

(d) $F(x) = \dfrac{x^2}{(x^2 - 9x + 14)(x^2 + 2x - 3)}$.

(e) $F(x) = \dfrac{x^2}{(x^2 - 9x + 14)(x^2 + 2x - 3)} - \dfrac{1}{(x^2 - 9x + 14)(x^2 + 2x - 3)}$.

[Bonus] **Problem 6.** Show that if p is a prime number different from 2 and 5, then it divides at least one of the elements of the set $\{1, 11, 111, 1111, \ldots\}$.