Discrete Structures EPFL, Fall 2014

Solution to Graded Problem Set 11

Date: 28.11.2014 Due date: 5.12.2014

Problem 1.

(a) This is equivalent to counting the number of subsets of $S \setminus \{5, 6\}$. Since $|S \setminus \{5, 6\}| = 8$, there are 2^8 choices.

(b) This is equivalent to counting the number of subsets of $\{2, 4, 6, 8, 10\}$. Since $|\{2, 4, 6, 8, 10\}| = 5$, there are 2^5 choices.

(c) There is only 1 subset which fulfills the requirements, i.e., $\{2, 4, 6, 8, 10\}$.

(d) First, let us count the number of subsets of 5 elements which include 3, but not 4. This number is equal to the number of subsets of 4 elements of $S \setminus \{3, 4\}$, which equals the number of combinations of 4 elements from a class of 8, i.e., $\binom{8}{4}$. Similarly, the number of subsets of 5 elements which include 4, but not 3, is $\binom{8}{4}$. Hence, the solution is $2\binom{8}{4}$.

(e) The sum of the elements of a subset of S is even if and only if the number of odd elements is even. Hence, we need to look for the subsets of 4 elements of S with either 0 or 2 or 4 even numbers. Hence, the solution is $\binom{5}{4}\binom{5}{0} + \binom{5}{2}\binom{5}{2} + \binom{5}{0}\binom{5}{4} = 2\binom{5}{4} + \binom{5}{2}\binom{5}{2}$.

Problem 2.

(a) We will prove here the following more general statement. Given any pair of positive integers n and k s.t. $k \leq n$, the number of distinct k-tuples of positive integers whose sum is n is given by the binomial coefficient $\binom{n-1}{k-1}$.

The number of solutions to the equation is equal to the number of ways in which we can place n balls into k boxes numbered from 1 to k, so that each box contains at least one ball. In short, configurations are only distinguished by the number of balls present in each box. Imagine to place the balls on a line and assume that the balls for the first box are taken from the left, followed by the balls for the second box, and so on. Clearly, the configuration will be determined once one knows the position on the line of the first ball going to the second box, the position of the first ball going to the third box, and so on. One can indicate this by placing $k - 1$ separating bars in $k - 1$ places between two balls. Since no box is allowed to be empty, there can be at most one bar between a given pair of balls. As a result, one views the n balls as fixed objects defining $n - 1$ gaps, in each of which there may or not be one bar. One has to choose $k - 1$ gaps to actually contain a bar. Therefore, there are $\binom{n-1}{k-1}$ possible configurations, which is also the number of solutions to the equation.

Eventually, the desired number of solutions is $\binom{31}{2}$.

(b) Let $x' = x + 1$, $y' = y + 1$, and $z' = z + 1$. Then, if $x + y + z = 32$, we have that

$$x' + y' + z' = 35.$$

In addition, if x, y, and z are non-negative integers, we have that x', y', and z' are positive integers. As a result, the number of non-negative triplets (x, y, z) that solve $x + y + z = 32$ is equal to the number of positive triplets (x', y', z') that solve $x' + y' + z' = 35$. Using the result previously proved, we have that the required number of solutions is equal to $\binom{34}{2}$.

(c) Let $x' = x - 6$, $y' = y - 14$, and $z' = z + 1$. Then, if $x + y + z = 32$, we have that

$$x' + y' + z' = 13.$$

In addition, if $x \geq 7$, $y \geq 14$, and $z \geq 0$, we have that x', y', and z' are positive integers. As a result, the number of desired solutions is equal to the number of positive triplets (x', y', z') that solve $x' + y' + z' = 13$. Using the result previously proved, we have that the required number of solutions is equal to $\binom{12}{2}$.

Problem 3. From the binomial theorem, we have that

$$(a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = a^p + b^p + \sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i}.$$

Now, we know that the binomial coefficient $\binom{p}{i}$ is an integer for any $p \in \mathbb{N}$ and any, $i \in \{0, 1, \cdots, p\}$. In addition, we have that

$$\binom{p}{i} = p \frac{(p-1)!}{i!(p-i)!}.$$

If $i \in \{1, 2, \cdots, p-1\}$, then $p > i$ and $p > p - i$. Therefore, since p is prime, $\gcd(p, i!(p-i)!) = 1$ and

$$\binom{p}{i} \equiv 0 \pmod{p}.$$

Consequently, we have that

$$\sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i} \equiv 0 \pmod{p}.$$

As a result, we finally obtain

$$(a+b)^p \equiv a^p + b^p \pmod{p}.$$

Problem 4.

Throughout the whole exercise, let $G(x) := \sum_{n=0}^{\infty} a_n x^n$ and note that for any $k \in \mathbb{N}$, $x^k G(x) = \sum_{n=k}^{\infty} a_{n-k} x^n$.

(a)

$$\begin{aligned} G(x) - 15xG(x) + 56x^2 G(x) &= \sum_{n=0}^{\infty} a_n x^n - 15 \sum_{n=1}^{\infty} a_{n-1} x^n + 56 \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= (a_0 + a_1 x - 15 a_0 x) + \sum_{n=2}^{\infty} (a_n - 15 a_{n-1} + 56 a_{n-2}) x^n \end{aligned}$$

Since the recurrence implies $a_n - 15a_{n-1} + 56a_{n-2} = 0$, plugging in the values of $a_0 = 0$ and $a_1 = 1$ we get

$$G(x)[1 - 15x + 56x^2] = x \qquad \iff \qquad G(x) = \frac{x}{1 - 15x + 56x^2}.$$

Decomposing $G(x)$ into partial fractions, we get

$$G(x) = \frac{1}{1-8x} - \frac{1}{1-7x}.$$

Using the power series we have

$$G(x) = \sum_{n=0}^{\infty} 8^n x^n - \sum_{n=0}^{\infty} 7^n x^n,$$

which implies that the solution of the recursion is

$$a_n = 8^n - 7^n.$$

(b)

$$\begin{aligned} G(x) - 2xG(x) - 15x^2G(x) &= \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n - 15\sum_{n=2}^{\infty} a_{n-2}x^n \\ &= (a_0 + a_1x - 2a_0x) + \sum_{n=2}^{\infty}(a_n - 2a_{n-1} - 15a_{n-2})x^n \end{aligned}$$

Since the recurrence implies $a_n - 2a_{n-1} - 15a_{n-2} = 0$, plugging in the values of $a_0 = 2$ and $a_1 = 2$ we get

$$G(x)[1 - 2x - 15x^2] = 2 - 2x \qquad \Longleftrightarrow \qquad G(x) = \frac{2-2x}{1-2x-15x^2}.$$

Decomposing $G(x)$ into partial fractions, we get

$$G(x) = \frac{1}{1+3x} + \frac{1}{1-5x}$$

Using the power series we have

$$G(x) = \sum_{n=0}^{\infty}(-3)^n x^n + \sum_{n=0}^{\infty} 5^n x^n,$$

which implies that the solution of the recursion is

$$a_n = (-3)^n + 5^n.$$

(c)

$$\begin{aligned} G(x) - 4xG(x) + 4x^2G(x) &= \sum_{n=0}^{\infty} a_n x^n - 4\sum_{n=1}^{\infty} a_{n-1}x^n + 4\sum_{n=2}^{\infty} a_{n-2}x^n \\ &= (a_0 + a_1x - 4a_0x) + \sum_{n=2}^{\infty}(a_n - 4a_{n-1} + 4a_{n-2})x^n \end{aligned}$$

Since the recurrence implies $a_n - 4a_{n-1} + 4a_{n-2} = 0$, plugging in the values of $a_0 = 1$ and $a_1 = 1$ we get

$$G(x)[1 - 4x + 4x^2] = 1 - 3x \qquad \Longleftrightarrow \qquad G(x) = \frac{1-3x}{1-4x+4x^2}.$$

Decomposing $G(x)$ into partial fractions, we get

$$G(x) = \frac{3/2}{1-2x} - \frac{1/2}{(1-2x)^2}$$

Using the power series we have

$$G(x) = \frac{3}{2}\sum_{n=0}^{\infty} 2^n x^n - \frac{1}{2}\sum_{n=1}^{\infty} n2^{n-1}x^{n-1} = \frac{3}{2}\sum_{n=0}^{\infty} 2^n x^n - \frac{1}{2}\sum_{n=0}^{\infty}(n+1)2^n x^n,$$

which implies that the solution of the recursion is

$$a_n = \frac{3}{2}\cdot 2^n - \frac{n+1}{2}\cdot 2^n = 2^n - \frac{n}{2}\cdot 2^n.$$

(d)

$$G(x) - 7xG(x) + 14x^2G(x) - 8x^3G(x) = \sum_{n=0}^{\infty} a_n x^n - 7\sum_{n=1}^{\infty} a_{n-1}x^n + 14\sum_{n=2}^{\infty} a_{n-2}x^n - 8\sum_{n=3}^{\infty} a_{n-3}x^n$$

$$= (a_0 + a_1x + a_2x^2 - 7a_0x - 7a_1x^2 + 14a_0x^2) + \sum_{n=3}^{\infty}(a_n - 7a_{n-1} + 14a_{n-2} - 8a_{n-3})x^n$$

Since the recurrence implies $a_n - 7a_{n-1} + 14a_{n-2} - 8a_{n-3} = 0$, plugging in the values of $a_0 = 3/4$, $a_1 = 1$, and $a_2 = 3$, we get

$$G(x)[1 - 7x + 14x^2 - 8x^3] = \frac{3}{4} - \frac{17}{4}x + \frac{13}{2}x^2 \qquad \Longleftrightarrow \qquad G(x) = \frac{\frac{3}{4} - \frac{17}{4}x + \frac{13}{2}x^2}{1 - 7x + 14x^2 - 8x^3}.$$

Decomposing $G(x)$ into partial fractions, we get

$$G(x) = \frac{1}{1-x} - \frac{1/2}{1-2x} + \frac{1/4}{1-4x}$$

Using the power series we have

$$G(x) = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty} 2^n x^n + \frac{1}{4}\sum_{n=0}^{\infty} 4^n x^n,$$

which implies that the solution of the recursion is

$$a_n = 1 - \frac{1}{2}\cdot 2^n + \frac{1}{4}\cdot 4^n.$$

(e)

$$G(x) - 2xG(x) - 5x^2G(x) + 6x^3G(x) = \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n - 5\sum_{n=2}^{\infty} a_{n-2}x^n + 6\sum_{n=3}^{\infty} a_{n-3}x^n$$

$$= (a_0 + a_1x + a_2x^2 - 2a_0x - 2a_1x^2 - 5a_0x^2) + \sum_{n=3}^{\infty}(a_n - 2a_{n-1} - 5a_{n-2} + 6a_{n-3})x^n$$

Since the recurrence implies $a_n - 2a_{n-1} - 5a_{n-2} + 6a_{n-3} = 0$, plugging in the values of $a_0 = 0$, $a_1 = -1$, and $a_2 = 11$, we get

$$G(x)[1 - 2x - 5x^2 + 6x^3] = -x + 13x^2 \qquad \Longleftrightarrow \qquad G(x) = \frac{\frac{3}{4} - \frac{17}{4}x + \frac{13}{2}x^2}{1 - 7x + 14x^2 - 8x^3}.$$

Decomposing $G(x)$ into partial fractions, we get

$$G(x) = -\frac{2}{1-x} + \frac{1}{1+2x} + \frac{1}{1-3x}$$

Using the power series we have

$$G(x) = -2\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(-2)^n x^n + \sum_{n=0}^{\infty} 3^n x^n,$$

which implies that the solution of the recursion is

$$a_n = -2 + (-2)^n + 3^n.$$

Problem 5.

(a) This is the same generating function of problem 4, part (a). Hence, the sequence is given by

$$a_n = 8^n - 7^n, \tag{1}$$

which is $\Theta(8^n)$.

(b) Let a_n be the sequence given by (1). This sequence has generating function $\frac{x}{1-15x+56x^2}$. Let b_n be the sequence with generating function $\frac{x^{32}}{1-15x+56x^2} = x^{31} \cdot \frac{x}{1-15x+56x^2}$. Then, $b_n = a_{n-31}$ for any $n \geq 31$. Therefore, b_n is $\Theta(a_n)$, which is $\Theta(8^n)$.

(c) The roots of the denominator are $x = 2$, $x = 7$, $x = -3$, and $x = 1$. Hence, decomposing $F(x)$ into partial fractions, we get

$$F(x) = \frac{A}{1 - \frac{1}{2}x} + \frac{B}{1 - \frac{1}{7}x} + \frac{C}{1 - x} + \frac{D}{1 + \frac{1}{3}x},$$

where A, B, C, D are non-zero. Therefore, the sequence has the form

$$A \cdot \left(\frac{1}{2}\right)^n + B \cdot \left(\frac{1}{7}\right)^n + C + D \cdot \left(-\frac{1}{3}\right)^n, \tag{2}$$

and it is $\Theta(1)$, since $C \neq 0$. Note that, in order to solve the exercise we do not need to know the values of A, B, C, and D, but it suffices to know that $C \neq 0$.

(d) Using the same argument as in point (b), we have that this sequence is the shifted version (by 2 positions) of the sequence defined in (2). Hence, it is $\Theta(1)$.

(e) The generating function can be re-written as

$$\frac{x^2 - 1}{(x^2 - 9x + 14)(x^2 + 2x - 3)} = \frac{x + 1}{(x^2 - 9x + 14)(x + 3)}.$$

By decomposing it into partial fractions, we get

$$F(x) = \frac{A'}{1 - \frac{1}{2}x} + \frac{B'}{1 - \frac{1}{7}x} + \frac{C'}{1 + \frac{1}{3}x},$$

where A', B', and C' are non-zero. Therefore, the sequence has the form

$$A' \cdot \left(\frac{1}{2}\right)^n + B' \cdot \left(\frac{1}{7}\right)^n + C' \cdot \left(-\frac{1}{3}\right)^n,$$

and it is $\Theta\left(\left(\frac{1}{2}\right)^n\right)$.

Problem 6. Since p is different from 2 and 5, we have that $10^{p-1} \equiv 1 \pmod{p}$, which implies that $p \mid 10^{p-1} - 1$. Now, note that

$$10^{p-1} - 1 = 9 \sum_{i=0}^{p-2} 10^i.$$

Hence, we have

$$p \mid 9 \sum_{i=0}^{p-2} 10^i.$$

Let us assume that p and 9 are coprime, i.e., $p \neq 3$. Then,

$$p \mid \sum_{i=0}^{p-2} 10^i = \overbrace{111\cdots1}^{p-1 \text{ times}}.$$

Since $\overbrace{111\cdots1}^{p-1 \text{ times}}$ belongs to the required set, we just proved the claim for $p \neq 3$. In addition, if $p = 3$, then it divides 111, which concludes the proof.