Discrete Structures EPFL, Fall 2014

Solution to Problem Set 10

Date: 21.11.2014 Not graded

Problem 1. Since we have 10 questions and 4 possible answers for each question in total we can have 4^{10} different answer sheets (using the product rule). Therefore, if we have $2 \times 4^{10} + 1$ students, using the pigeonhole principle we can conclude that there are at least three identical answer sheets.

Problem 2. First of all you can simply check that the answer is obviously wrong because $\binom{26}{3} \times \binom{49}{2} > \binom{52}{5}$ which represents the total number of possible ways we can pick a 5-cards hand.

Now to see why the answer is wrong consider one possible 5-cards hand consisting of the A, K, Q, J, and 10 of say hearts. This hand is being counted at least twice. It is being counted once by picking A, K, and Q in first $\binom{26}{3}$ choices and then J and 10 among the remaining $\binom{49}{2}$ cards and another time by considering 10, J, and Q in first $\binom{26}{3}$ choices and then K and A among the remaining $\binom{49}{2}$.

The correct way of counting the number of 5-card hands containing at least 3 red cards is as follows: Our 5-card hands can contain 3, 4, or 5 red cards. Hence the total number of desired 5-card hands is

$$\binom{26}{3} \times \binom{26}{2} + \binom{26}{4} \times \binom{26}{1} + \binom{26}{5}.$$

Problem 3.

(a) Clearly, $E_i \subset E$ for any $i \in \{0, 1, \cdots, n\}$. Therefore $E_0 \cup E_1 \cup \cdots \cup E_n \subset E$. Also, $E \subset E_0 \cup E_1 \cup \cdots \cup E_n$, because every string has a number of 1's which is an integer between 0 and n. Consequently, $E = E_0 \cup E_1 \cup \cdots \cup E_n$.

(b) Counting the bit strings with exactly i 1's is equivalent to counting the ways in which one can choose the positions of these i 1's. These are the combinations of i elements from a class of n without replacement. Therefore, $E_i = \binom{n}{i}$.

(c) $|E_i \cap E_j| = 0$ for $i \neq j$, because a binary string cannot have simultaneously i and j 1's with $i \neq j$.

(d) The number of binary strings of length n, i.e., the cardinality of E, is equal to 2^n. In addition, $|E| = \sum_{i=0}^{n} |E_i|$, because the sets E_i for $i \in \{0, 1, \cdots, n\}$ are pairwise disjoint by point (c). Using this last equation and the result of point (b), we obtain that

$$2^n = |E| = \sum_{i=0}^{n} |E_i| = \sum_{i=0}^{n} \binom{n}{i}.$$

Problem 4.

(a) Recall that $\binom{m}{n} = \dfrac{m!}{n!(m-n)!}$. Then, for any $m \geq 1$ and any $1 \leq n \leq m-1$,

$$\begin{aligned}\binom{m-1}{n} + \binom{m-1}{n-1} &= \frac{(m-1)!}{n!(m-1-n)!} + \frac{(m-1)!}{(n-1)!(m-n)!} \\ &= \frac{(m-1)!(m-n+n)}{n!(m-n)!} = \frac{m!}{n!(m-n)!} = \binom{m}{n}.\end{aligned}$$

(b) The total number of ways we can pick m balls and color them black is $\binom{m}{n}$. The other way for counting this is to consider a particular ball, if we decide to color this black, we need to color $n-1$ other balls among remaining $m-1$ to as well for which have $\binom{m-1}{n-1}$ ways. Otherwise, i.e., if we leave that particular ball white, we need to pick n among remaining $m-1$ to color. This can be done in $\binom{m-1}{n-1}$ ways.

(c) **Base Step.** For $n = 0$, $\sum_{i=0}^{n} \binom{n}{i} = 1$ and $2^n = 1$.

Inductive Step. Assume that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. Then, using the identity (2) and the induction hypothesis, we obtain that

$$\begin{aligned}\sum_{i=0}^{n+1} \binom{n+1}{i} &= \binom{n+1}{0} + \binom{n+1}{n+1} + \sum_{i=0}^{n-1} \binom{n+1}{i+1} = 2 + \sum_{i=0}^{n-1} \binom{n}{i} + \sum_{i=0}^{n-1} \binom{n}{i+1} \\ &= \binom{n}{n} + \binom{n}{0} + \sum_{i=0}^{n-1} \binom{n}{i} + \sum_{i=1}^{n} \binom{n}{i} = 2\sum_{i=0}^{n} \binom{n}{i} = 2 \cdot 2^n = 2^{n+1}.\end{aligned}$$

Problem 5.

(a) $a_n = a_{n-1}/1.03$ with initial condition $a_0 = 1$.

(b) $a_{20} = 1/1.03^{20} \approx 0.55$.

(c) In general, $a_n = 1/1.03^n$. Then,

$$a_n \leq 0.1 \iff n \geq \frac{\log 10}{\log 1.03} = 77.9.$$

Therefore, after 78 years the purchasing power is ≤ 0.1.

(d) If the inflation is ten percent annually, the recursion is $b_n = b_{n-1}/1.1$ with initial condition $b_0 = 1$. Since $b_{20} \approx 0.15$ and $a_{80} = 0.09$, then it is better to suffer from an inflation of ten percent annually for 20 years, because in this case the purchasing power is higher.

Problem 6.

(a) The following passages are sufficient to prove the claim:

$$4a_{n-1} - 3a_{n-2} = 4 \cdot (\alpha 3^{n-1} + \beta) - 3 \cdot (\alpha 3^{n-2} + \beta) = \alpha(4 \cdot 3^{n-1} - 3^{n-1}) + \beta = \alpha 3^n + \beta = a_n.$$

(b) Note that $a_0 = \alpha + \beta$ and $a_1 = 3\alpha + \beta$. Then, by solving the system

$$\begin{cases} \alpha + \beta = \pi \\ 3\alpha + \beta = \pi + 3\sqrt{2} \end{cases},$$

we obtain that

$$\begin{cases} \alpha = \frac{3\sqrt{2}}{2} \\ \beta = \pi - \frac{3\sqrt{2}}{2} \end{cases}.$$

Hence, the solution of the recurrence is $a_n = \frac{3\sqrt{2}}{2}3^n + \pi - \frac{3\sqrt{2}}{2}$.

(c) Let $G(x) := \sum_{n=0}^{\infty} a_n x^n$ and note that for any $k \in \mathbb{N}$, $x^k G(x) = \sum_{n=k}^{\infty} a_{n-k} x^n$. Then,

$$\begin{aligned} G(x) - 4xG(x) + 3x^2G(x) &= \sum_{n=0}^{\infty} a_n x^n - 4\sum_{n=1}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n \\ &= (a_0 + a_1 x - 4a_0 x) + \sum_{n=2}^{\infty}(a_n - 4a_{n-1} + 3a_{n-2})x^n \end{aligned}$$

Since the recurrence implies $a_n - 4a_{n-1} + 3a_{n-2} = 0$, plugging in the values of $a_0 = \pi$ and $a_1 = \pi + 3\sqrt{2}$ we get

$$G(x)[1 - 4x + 3x^2] = [\pi + (3\sqrt{2} - 3\pi)x] \qquad \iff \qquad G(x) = \frac{\pi + 3(\sqrt{2} - \pi)x}{1 - 4x + 3x^2}$$

Decomposing $G(x)$ into partial fractions we get

$$G(x) = \frac{\pi - 3\sqrt{2}/2}{1 - x} + \frac{3\sqrt{2}/2}{1 - 3x}$$

Using the power series we have

$$G(x) = (\pi - 3\sqrt{2}/2)\sum_{n=0}^{\infty} x^n + 3\sqrt{2}/2\sum_{n=0}^{\infty} 3^n x^n$$

which implies that the solution of the recursion is

$$a_n = (\pi - \frac{3\sqrt{2}}{2}) + \frac{3\sqrt{2}}{2}3^n.$$