ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 21 Information Theory and Coding
Homework 9 November 19, 2013

PROBLEM 1. Let $\{f_i : \mathbb{R} \to \mathbb{R}\}_{1\leq i\leq n}$ be a set of convex functions on $\mathbb{R}$ and $c_i \geq 0$ for all $i \in \{1, 2, \dots, n\}$.

(a) Show that the function $f : x \mapsto \sum_{i=1}^{n} c_i f_i(x)$ is convex.

(b) Show that the function $g : (x_1, x_2, \dots, x_n) \mapsto \sum_{i=1}^{n} c_i f_i(x_i)$ is convex.

PROBLEM 2. Let $\{f_i(x)\}_{i\in I}$ be a set of convex real-valued functions defined over D. Assuming that $f(x) = \sup_{i\in I} f_i(x)$ is finite for all $x \in D$, show that $f(x)$ is convex.

PROBLEM 3. Let $f : U \to \mathbb{R}$ be a convex function on U and assume that there exists $a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b$ for all $x \in U$. Let h be an increasing convex function defined on the interval $[a, b]$. Show that the function $g = h \circ f$ is convex on U.

PROBLEM 4. A function $f(v)$ is defined on a convex region R of a vector space. Show that $f(v)$ is convex iff the function $f(\lambda v_1 + (1 - \lambda)v_2)$ is a convex function of λ, $0 \leq \lambda \leq 1$, for all $v_1, v_2 \in R$.

PROBLEM 5. Let $\mathcal{X}$ and $\mathcal{Y}$ be the input and output alphabets, respectively, of a discrete memoryless channel.

(a) Show that $H(Y)$ is a concave function of the input probability vector.
Hint: Consider the output probability vectors resulting from the input probability vectors.

(b) Give an example of a discrete memoryless channel for which the concavity of $H(Y)$ is not strict.

(c) Show that $-H(Y|X)$ is a linear function of the input probability vector.

(d) Combining (a) and (c), show that $I(X;Y)$ is a concave function of the input probability vector.

PROBLEM 6. Holder's inequality. Let $\{a_i, b_i\}_{1\leq i\leq n}$ be a set of non-negative real numbers, $n \in \mathbb{N}$, and let $\lambda \in (0, 1)$. Show that

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^{1/\lambda}\right)^{\lambda} \left(\sum_{i=1}^{n} b_i^{1/(1-\lambda)}\right)^{1-\lambda}$$

with equality iff there exists some c that satisfies $a_i^{1-\lambda} = b_i^{\lambda} c$ for all $i \in \{1, 2, \dots, n\}$. What is the special case $\lambda = \frac{1}{2}$?
Hint: Define

$$Q_i = \frac{a_i^{1/\lambda}}{\sum_{i=1}^{n} a_i^{1/\lambda}}$$

and

$$P_i = \frac{b_i^{1/(1-\lambda)}}{\sum_{i=1}^{n} b_i^{1/(1-\lambda)}}$$

and observe that they are non-negative numbers which sum to one. Then, use the convexity of $\lambda \mapsto \sum_{i=1}^{n} Q_i^{\lambda} P_i^{1-\lambda}$ to show $\sum_{i=1}^{n} Q_i^{\lambda} P_i^{1-\lambda} \leq 1$ with equality iff, for all $i \in \{1, 2, \dots, n\}$, $P_i = Q_i$.