ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6 Information Theory and Coding

Homework 3 Oct. 01, 2013

Problem 1. Let $p_{XY}(x, y)$ be given by

X \ Y	0	1
0	1/3	1/3
1	0	1/3

Find

(a) $H(X)$, $H(Y)$.

(b) $H(X|Y)$, $H(Y|X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X; Y)$.

(f) Draw a Venn diagram for the quantities in (a) through (e).

Problem 2. Let X be a random variable taking values in M points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) H(Y)$$

where Y is a random variable taking values in $M - 1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1 - \alpha)$; $1 \le j \le M - 1$. Show that

$$H(X) \le \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)$$

and determine the condition for equality.

Problem 3. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X, Y; Z) \ge I(X; Z)$.

(b) $H(X, Y|Z) \ge H(X|Z)$.

(c) $H(X, Y, Z) - H(X, Y) \le H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \ge I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

Problem 4. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1})$.

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n | X_{n-1}, \ldots, X_1)$.

Problem 5. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0 | X_{-1}, \ldots, X_{-n}) = H(X_0 | X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

Problem 6. Show, for a Markov chain, that

$$H(X_0 | X_n) \geq H(X_0 | X_{n-1}), \quad n \geq 1.$$

Thus, initial state X_0 becomes more difficult to recover as time goes by.

Problem 7. Let $X_1, X_2, \ldots$ be i.i.d., each with probability distribution $p(x)$. Show that with probability one

$$\lim_{n \to \infty} p(X_1, \ldots, X_n)^{1/n}$$

exists, and find its value. Hint: use the AEP.

Compare this to

$$\lim_{n \to \infty} E[p(X_1, \ldots, X_n)^{1/n}].$$