ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 10 Information Theory and Coding
Homework 5 Oct. 16, 2012

PROBLEM 1. Define the *type* $P_{\mathbf{x}}$ (or empirical probability distribution) of a sequence $x_1, \ldots, x_n$ be the relative proportion of occurences of each symbol of $\mathcal{X}$; i.e., $P_{\mathbf{x}}(a) = N(a|\mathbf{x})/n$ for all $a \in \mathcal{X}$, where $N(a|\mathbf{x})$ is the number of times the symbol a occurs in the sequence $\mathbf{x} \in \mathcal{X}^n$.

(a) Show that if $X_1, \ldots, X_n$ are drawn i.i.d. according to $Q(x)$, the probability of $\mathbf{x}$ depends only on its type and is given by

$$Q^n(\mathbf{x}) = 2^{-n(H(P_{\mathbf{x}})+D(P_{\mathbf{x}}||Q))}.$$

Hint: Start by showing the following:

$$Q^n(\mathbf{x}) = \prod_{i=1}^{n} Q(x_i) = \prod_{a \in \mathcal{X}} Q(a)^{N(a|\mathbf{x})} = \prod_{a \in \mathcal{X}} Q(a)^{nP_{\mathbf{x}}(a)}$$

Define the type class $T(P)$ as the set of sequences of length n and type P:

$$T(P) = \{\mathbf{x} \in \mathcal{X}^n : P_{\mathbf{x}} = P\}.$$

For example, if we consider binary alphabet, the type is defined by the number of 1's in the sequence and the size of the type class is therefore $\binom{n}{k}$.

(b) Show for a binary alphabet that

$$|T(P)| \doteq 2^{nH(P)}. \tag{1}$$

We say that $a_n \doteq b_n$, if $\lim_{n\to\infty} \frac{1}{n} \log \frac{a_n}{b_n} = 0$.

Hint: Prove that

$$\frac{1}{n+1} 2^{nh_2(\frac{k}{n})} \leq \binom{n}{k} \leq 2^{nh_2(\frac{k}{n})}.$$

$h_2(\cdot)$ *denotes the binary entropy function. To derive the upper bound, start by proving*

$$1 \geq \binom{n}{k} \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k} = \binom{n}{k} 2^{n\left(\frac{k}{n}\log\frac{k}{n} + \frac{n-k}{n}\log\frac{n-k}{n}\right)}.$$

To derive the lower bound, start by proving

$$1 = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \leq (n+1) \max_j \binom{n}{j} p^j (1-p)^j,$$

then take $p = k/n$ and show that the maximum occurs for $j = k$.

(c) Use (a) and (b) to show that

$$Q^n(T(P)) \doteq 2^{-nD(P||Q)}.$$

Note: $D(P||Q)$ is the informational divergence (or Kullback-Leibler divergence) between two probability distributions P and Q on a common alphabet $\mathcal{X}$ and is defined as

$$D(P||Q) = \sum_{a \in \mathcal{X}} P(a) \log \frac{P(a)}{Q(a)}.$$

Recall that we have already seen the non-negativity of this quantity in the class.

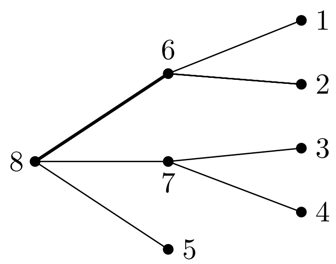
PROBLEM 2. Construct a Tunstall code with $M = 8$ words in the dictionary for a binary memoryless source with $P(0) = 0.9$, $P(1) = 0.1$.

PROBLEM 3. Consider a valid, prefix-free dictionary of words from a source of alphabet size D. Show that the set of lengths $L_1, \dots, L_M$ of the dictionary words satisfy the Kraft inequality

$$\sum_j D^{-L_j} \leq 1$$

with equality. Show that if the dictionary is valid, but not prefix-free, then the Kraft inequality is violated.

PROBLEM 4. Consider a tree with M leaves $n_1, \dots, n_M$ with probabilities $P(n_1), \dots, P(n_M)$. Each intermediate node n of the tree is then assigned a probability $P(n)$ which is equal to the sum of the probabilities of the leaves that descend from it. Label each branch of the tree with the label of the node that is on that end of the branch further away from the root. Let $d(n)$ be a "distance" associated with the branch labelled n. The distance to a leaf is the sum of the branch distances on the path to from root to leaf.



For example, in the tree shown above, nodes 1, 2, 3, 4, 5 are leaves, the probability of node 6 is given by $P(1) + P(2)$, the probability of node 7 by $P(3) + P(4)$, of node 8 (root) by $P(1) + P(2) + P(3) + P(4) + P(5) = 1$. The branch indicated by the heavy line would be labelled 6. The distance to leaf 2 is given by $d(6) + d(2)$.

(a) Show that the expected distance to a leaf is given by $\sum_n P(n)d(n)$ where the sum is over all nodes other than the root. Recall that we showed this in the class for $d(n) = 1$.

(b) Let $Q(n) = P(n)/P(n')$ where n' is the parent of n, and define the entropy of an intermediate node n' as

$$H_{n'} = \sum_{n\colon n \text{ is a child of } n'} -Q_n \log Q_n.$$

Show that the entropy of the leaves

$$H(\text{leaves}) = -\sum_{j=1}^{M} P(n_j) \log P(n_j)$$

is equal to $\sum_{n \in I} P(n)H_n$ where the sum is over all intermediate nodes including the root. Hint: use part (a) with $d(n) = -\log Q(n)$.

(c) Let X be a memoryless source with entropy H. Consider some valid prefix-free dictionary for this source and consider the tree where leaf nodes corresponds to dictionary words. Show that $H_n = H$ for each intermediate node in the tree, and show that

$$H(\text{leaves}) = E[L]H$$

where $E[L]$ is the expected word length of the dictionary. Note that we proved this result in class by a different technique.

PROBLEM 5. Suppose we bet our fortune at a casino game which multiplies our fortune by a random variable X, with

$$\Pr(X = 1/4) = \Pr(X = 2) = 1/2.$$

We play the game repeatedly, starting with an initial fortune $F_0 = 1$, betting our entire fortune at each time. The value of X at the ith game is denoted by X_i. These values are independent and distributed as X.

(a) What is the expected value $f_n = E[F_n]$, our fortune after n plays? How does f_n behave as n gets large?

(b) What is $l_n = E[\log_2 F_n]$? How does 2^{l_n} behave as n gets large?

(c) Does F_n concentrate around f_n or 2^{l_n}? [Hint: does the law of large numbers apply to F_n or to $\log_2 F_n$?]

(d) With this 'bet all we have' strategy do we get rich or poor?

(e) If we had kept a fraction r of our fortune in reserve at each play, could we have done better? What is the best value of r to maximize $\lim_n \frac{1}{n} \log_2 F_n$, the 'rate of growth' of fortune.