ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Midterm **Graph Theory Applications**

Date: April 11, 2013 Spring 2013

The midterm exam is *closed* book, but you can use one A4 piece of paper where you can note anything you want. The exam lasts from 4:15pm till 6pm. DON'T PANIC. It is not necessarily expected that you solve all problems. Solve those problems first that you find easiest and then gradually move to the harder ones. Be concise! The solution for each of the four problems can be written down in a matter of some lines (not pages). We will subtract points for any material in your answer is not directly related to the solution.

Problem 1 (20pts)**.** Let G be a k-regular bi-partite graph on the vertex set $V = X \cup Y$, $|X| = |Y|$, and edge set E. Let A be the adjacency matrix corresponding to G. Show that A has an eigenvalue of k as well as an eigenvalue of $-k$. What are the two eigenvectors which correspond to these two eigenvalues?

Problem 2 (20pts)**.** Given a simple connected graph $G = (V, E)$ with edge costs w_e for each $e \in E$ (assume all edge costs are distinct), prove the following two fundamental properties that are used in all Minimum Spanning Tree (MST) algorithms.

a) (Cut Property) For any proper subset $S \subset V$ of nodes in G, let $e = (u, v)$ be the edge with minimum weight such that $u \in S$ and $v \in V \setminus S$. Show that every MST must contain e.

b) (Cycle Property) Let C be a cycle in G. Let $e = (u, v)$ be the edge with maximum weight in C. Show that e is not in any MST of G.

Problem 3 (20pts)**.** Let G be a bipartite graph, with bipartition (X, Y) with no isolated vertices. Suppose that for every edge (x, y) with one end $x \in X$ and another end $y \in Y$, we have $\deg(x) \geq \deg(y)$. Prove that G has a matching that covers X.

Problem 4 (20pts)**.** [Who is afraid of directed graphs?] A *catenym* is a pair of words separated by a period such that the last letter of the first word is the same as the first letter of the second. For example, the following are catenyms: dog.gopher, gopher.rat, rat.tiger, etc. A *compound catenym* is a sequence of three or more words separated by periods such that each adjacent pair of words forms a catenym. For example, dog.gopher.rat.tiger is a compound catenym. Now, you are given a dictionary of words in English as input. Can you give an efficient procedure to determine if there is a compound catenym that uses each word in the dictionary exactly once?