ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Homework 2 **Graph Theory Applications**
Date: May 14, 2013 Spring 2013

Problem 1. A vertex with the property that when it is deleted along with the edges incident on it, the graph becomes disconnected is called a *cut vertex*. Let G be a regular graph of degree k with a cut-vertex v. Consider the k edges $(v_i v)$, $1 \leq i \leq k$ incident on v (colored respectively with color i) and assume that $\chi'(G) = \Delta(G) = k$. We will show that under this assumption, in fact v_i and v_j are connected even if we delete v from G, contradicting the nature of the vertex v.

Consider a proper k edge coloring of G and look at the vertices v_i and v_j in $G - \{v\}$. Starting with vertex v_i, find a maximum walk whose edges are alternately colored i and j. Since the edge connected to v_i with color i has been deleted in $G - \{v\}$, the walk has to start with color j. Along the walk, each vertex has exactly one edge with color i and one with color j. Note that we cannot repeat a vertex along the walk as that would either contradict the fact that we have an edge coloring or end up back in v_i which is not possible as the only edge adjacent to v_i with color i has been deleted. Thus, our walk ends when we reach a vertex that does not have any edge with color j. v_j is the only such vertex in $G - \{v\}$, which means we have found a path connecting v_i and v_j in $G - \{v\}$. Since this is true for all v_i,v_j, v cannot be a cut vertex of G, which is a contradiction.

Problem 2. Given the bipartite graph $G = (U, V; E)$, add a source and sink and connect all vertices in U to the source with capacity 1 edges. Connect all vertices of V to the sink with capacity 1 edges. Set the capacity of all edges in G to infinity. Now, we just need to show that any flow of value k in this new graph corresponds to a matching of size k. For the cuts, any finite value cut has to avoid the infinite capacity edges and every cut of size k corresponds to a vertex cover of size k. Finally, the max-flow min-cut theorem implies that the sizes of the maximum matching and minimum vertex cover are equal in a bipartite graph.

Problem 3. We will have a source s, a sink t, and a bipartite graph with n nodes on one side (representing the families) and m nodes on the other (representing the cars). Connect source to family node i with an edge of capacity F_i. Connect car node j to t with an edge of capacity C_j. Finally, connect each family node with each car node with a capacity 1 edge. The correspondence between the feasibility of a car sharing arrangement and the feasibility of a flow of value $\sum_{i=1}^{n} F_i$ is clear.

Problem 4. Create a graph with two processor nodes p_1 and p_2 and n module nodes $m_1, ldots, m_n$. Connect p_1 with m_i with an edge of cost a_i and connect p_2 with m_i with an edge of cost b_i. For each pair of modules m_i and m_j add an edge of cost c_{ij}. Then the best partition is given by the min-cut separating p_1 and p_2.