ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Homework 1 — **Graph Theory Applications**

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Problem 1. We construct a graph where people become vertices and there is an edge between two people if and only if these people are sitting next to each other during a dinner. From the problem statement, anybody can sit next to anybody, therefore our graph is the complete graph on $2k+1$ vertices. We want to find circular arrangements of the people where everybody appears exactly once. This corresponds to finding a Hamiltonian cycle in K_{2k+1}. Finally, we are looking for k such arrangements such that no person sits next to the same person in any two of these arrangements, so in graph theoretic terms, we are looking for k edge-disjoint Hamiltonian cycles in K_{2k+1}. We label the people by $0, 1, \ldots, 2k-1, \infty$.

Then the first cycle is

$$C_0 = \infty, 0, 2k-1, 1, 2k-2, 2, 2k-3, \ldots, k-1, k, \infty.$$

We get a new cycle by adding $0 \leq \ell \leq k-1$ to every element (except for ∞) and taking the mod $2k$ result:

$$C_\ell = \infty, \ell, 2k-1+\ell, 1+\ell, \ldots, k-1+\ell, k+\ell, \infty.$$

Clearly we get a Hamiltonian cycle for every ℓ. We need to show that these cycles are edge-disjoint. In other words, given an edge (i, j) we have to be able to uniquely identify the cycle in which the given edge appears.

For edges of the form (∞, i), note that for different values of ℓ both the first and the last edges of the cycles connect different vertices to vertex ∞. Also, the first edge connects only neighbors $i < k$, while the last connects neighbors $i \geq k$. Hence, for such an edge we can immediately see this property.

For edges (i, j), $i, j \neq \infty$, observe that the differences between the elements of the cycles (not looking at the first and the last edges) are $2k-1, 2, 2k-3, 4, \ldots, 1$. All these values are different (odd elements are odd and decreasing, even elements are even and increasing) an they are the same in every cycle. Hence, from computing $j-i$, we can tell at which position this pair of vertices has to appear in the cycle. But we also see that in a given position every cycle has a different element, so given an edge (i, j) we can uniquely tell in which cycle the given edge appears. We further have to note that it cannot happen that we find both (i, j) and (j, i), because the positions corresponding to $i-j$ and $j-i$ are complementary in the sense that what appears in one does not show up in the other.

One can visualize this construction by placing the vertices around a circle and placing ∞ in the middle. Adding $1 \mod 2k$ corresponds to a counter-clockwise rotation of the vertices while keeping the edges in place. As an example the construction is shown for $k = 4$, $\ell = 0$.

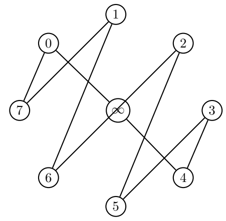

Figure 1: Construction for Problem 1

Problem 2. Let $\Delta(G) = \Delta$. If Δ is an eigenvalue of A, then we know that there exists a v for which

$$v(A - \Delta I) = 0. \tag{1}$$

We can also assume that the largest (absolute value) element of v is 1. (If not, we divide the vector by its largest element.) Let 1 be the i-th element of v. Consider the i-th column of $A' = A - \Delta I$. We know that the diagonal elements of A' are $-\Delta$, so we have from (1) that

$$\sum_{j=1, j\neq i}^{n} v_j {A'}_{j,i} = \Delta. \tag{2}$$

Also, we know that $\sum_{j=1,j\neq i}^{n} {A'}_{j,i}$ gives the degree of the ith vertex, so

$$\sum_{j=1, j\neq i}^{n} {A'}_{j,i} \leq \Delta. \tag{3}$$

Since v_i is the largest element in v, we must have that (3) is equality and $v_j = 1$, if $A'_{j,i} \neq 0$. In other words, every vertex j that is connected to i has $v_j = 1$.

We can repeat the same argument for v_j, and continuing the process, in the end for all elements of v, since G is connected. We see that every element of v equals 1, which implies that every column of A sums up to Δ, i.e. the graph is regular.

Problem 3. Let $C = \{c_1, c_2, \ldots, c_n\}$ be the n distinct colors and $D = \{d_1, \ldots, d_n\}$ the n distinct diameters. Create a bipartite graph $G(C \cup D, E)$, where edge (c_i, d_j) is in E if and only if there is a ball of color c_i and diameter d_j among the nk balls. Observe that the final graph G will be a k-regular bipartite graph, since every vertex c_i must have degree k and every vertex d_j must have degree k. By a proposition in the class notes, we conclude that this graph has a perfect matching M (of size n). Our final selection of balls is exactly the n balls corresponding to the edges in M: since the edges in M cover all vertices in C and all vertices in D, the corresponding balls cover all possible colors and all possible diameters.

Problem 4. We construct the following bipartite graph $G(X \cup Y; E)$, where vertices on the left hand side correspond to kids and vertices on the right hand side correspond to toys.

- $X = X_1 \cup X_2$ where $X_1 = \{x_1, \ldots, x_{15}\}$ corresponds to the set of 15 kids, and $X_2 = \{x'_1, \ldots x'_{15}\}$ is a copy of X_1 and we can think of the x'_is as clones of the kids;

- $Y = \{y_1, \ldots, y_30\}$ corresponds to the set of toys; and
- there is an edge $(x_i; y_j)$ if and only if kid i names toy j as one of its preferred toys. For every edge $(x_i; y_j)$, we also place an edge $(x_i'; y_j)$.

Observe that in G

1. For any set $S \subseteq X_1$ (or X_2), $N(S) \geq 2|S|$ by the teacher's claim.
2. By construction, for every $T' \subseteq X_2$, $N(T') = N(T)$.

Consider a set $S \cup T' \subseteq X_1 \cup X_2$. We have

$$\begin{aligned} N(S \cup T') &= N(S \cup T) && (4) \\ &\geq 2|S \cup T| && (5) \\ &= |S| + |T| + |S \setminus T| + |T \setminus S| && (6) \\ &\geq |S| + |T| && (7) \\ &= |S| + |T'| && (8) \\ &= |S \cup T'|, && (9) \end{aligned}$$

where the last equality follows from $S \setminus T = \emptyset$. Therefore G has a perfect matching which, as we know, can be computed efficiently. Let $(x_i; y_k)$ and $(x_i'; y_l)$ be edges appearing in the perfect matching: then give to kid i the k-th and the l-th toy.

Problem 5. Consider the bipartite graph $G(V_1 \cup V_2, E)$ where piles 1 up to 13 become the vertices of the left-hand side (V_1) and ranks become the vertices of the right-hand side (V_2), hence each side consists of 13 vertices. Edge (i, j) is in E if and only if there is (at least) one card of rank j in column i, one edge (i, j). Suppose that this graph has a perfect matching M. Form the following set of cards: for $1 \leq i \leq 13$, pick from pile i the (or one of the) card(s) that has rank j, where the edge (i, j) is in M. Clearly such a selection guarantees that one card is selected from each pile and that all of the selected cards have different ranks. It remains to show that G has a perfect matching. We will use Hall's theorem. Consider any set $S \subseteq V_1$. Then $N(S)$ is the set of ranks that the piles corresponding to the vertices in S contain. Now $|S|$ piles contain $4|S|$ cards, and these cards must contain at least $|S|$ different ranks (a deck contains exactly 4 cards of each rank!). Therefore $N(S) = |S|$. Since $|V_1| = |V_2|$, G has a perfect matching by Hall's theorem.

Problem 6. Consider a regular bipartite graph on $2n$ vertices, and degree k. Let e be any edge in this graph. We want to show that e can be included in some perfect matching. We will use induction on the degree k of the graph.
Base case: For $k = 1$ all edges in the graph are included in the perfect matching, therefore e is included.
Induction hypothesis: Assume that for $k - 1$, every edge of the $(k - 1)$-regular bipartite graph on $2n$ vertices can be included in a perfect matching.
Inductive step: Consider a k-regular bipartite graph on $2n$ vertices. Fix some edge e in the graph. By Hall's theorem, we know that there is a perfect matching in this graph. If e is included in the

perfect matching we are done. Otherwise, delete all edges of the perfect matching from the graph. The remaining graph G' is a $(k-1)$-regular bipartite graph, and by the induction hypothesis, every edge of G' can be included in a perfect matching for G', and therefore so can e. Since every PM for G' is a PM for G too (because $G' \subset G$), we conclude that e can be included in some PM for G.

Problem 7. We will prove the problem by induction on $n-k$. If $n-k=0$ then $n=k$ and there is nothing to prove. For $n-k=1$ we proceed as follows. Since $n-k=1$, we have $k=n-1$. This means that at each row or column of the table, $n-1$ cells are filled and only one cell is left empty. We fill all the blank cells with number n. Clearly that assignment satisfies the condition of the problem. Now suppose that if $n-k=m$ we can fill the blank cells appropriately. i.e we can put numbers $k+1, k+2, \ldots, n$ in the blank cells such that at each row and each column there exists precisely one number i, for $i=1,2,3,\ldots,n$. We show that if $n-k=m+1$, the same conclusion holds. The idea is to choose n free cells of the table so that no two of them are in the same row or column. If we can show that we can always find such cells, then we fill them with number $k+1$. The resulting table satisfies the condition of the problem. Moreover we have $n-(k+1)=n-k-1=m$ and therefore by induction, we can complete the rest of the table. Thus, we only need to show that we can find n blank cells such that no two are in the same row or column. Let G be a bipartite graph with the vertex set $A \cup B$ where: $A=\{a_1,a_2,\ldots,a_n\}, B=\{b_1,b_2,\ldots,b_n\}$ and the edge set: $E=\{(a_i,b_j)$: The cell located in the i-th row and the j-th column of the table is blank $\}$. Since at each row and column of the table, there are exactly $n-k$ blank cells, G is an r-regular bipartite graph and has a perfect matching. This perfect matching corresponds to n free cells such that now two of them are in the same row or column and therefore we are done.