ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Homework 1 **Graph Theory Applications**

Date: March 21, 2013 Spring 2013

This is the first graded homework. The problems are similar to what you can expect in the exams. In fact, some of these problems are old exam questions. As always you can talk with any of your colleagues about the problems. But each of you has to write down the solution on her own. In addition, if you have discussed any problems with colleagues or if you have used any other resources (such as Google!), write this down on top of your answer sheet. This is standard scholarly practice. If we discover similarities of solutions beyond what was indicated, you will receive 0 points for this homework. We will not investigate who copied from whom. NOTE: The midterm exam is *closed* book, but you can use one A4 piece of paper where you can note anything you want.

Problem 1. A set of $2k + 1$ people plan to have dinner together around a circular table on k successive nights in such a way that no pair sit next to each other more than once. Show that this is possible for $2k + 1 = 9$. Optionally, can you generalize this for any k?

Problem 2. We have seen in the class that if G is a k-regular graph, then its adjacency matrix A has k as eigenvalue. Now, prove the other direction: if G is a simple connected graph and its adjacency matrix A has $\Delta(G)$ as eigenvalue, then G is regular. ($\Delta(G)$ is the maximum degree).

Problem 3. Consider a set S of nk balls, each of which has one of n distinct colors and one of n distinct diameters. Furthermore, there are exactly k balls of each color and exactly k balls of each diameter. Show that n balls can be selected such that every diameter and color appears in the final selection.

Problem 4. In a small kindergarten, there are 15 kids and only 30 toys. Each morning, each kid names a set of favorite toys which (s)he wants to play with, and expects to get two of them. Distributing the toys is a big problem for the kindergarten teacher! If she assures you that for any group S of kids, the number of toys named by the kids in S is at least $|2S|$, can you translate her problem in graph-theoretic terms and give her a solution?

Problem 5. A deck of playing cards is arranged into thirteen piles each of which consists of 4 cards. Show how you can select one card from each column so that all of your selected cards have different ranks (i.e., you selected exactly one ace, one 2, one 3, etc.).

Problem 6. Prove that any edge in a regular bipartite graph can be included in some perfect matching.

Problem 7. Consider an $n \times n$ table consisting n^2 cells in which some of them are filled by some integer numbers. Let $k < n$ be an integer number and assume that for each $i = 1, 2, \ldots, k$, precisely n cells are filled by number i so that at each row and column there exists exactly one i. Prove that we can fill all the $n^2 - nk$ blank cells with the numbers $k + 1, k + 2, \ldots, n$ so that in the resulting table, in each row and column, all the numbers $1, 2, 3, \ldots, n$ appear.