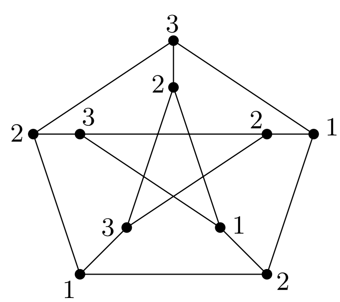
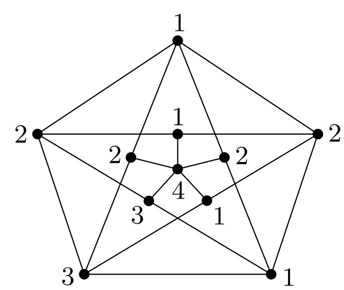
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Exercise 8 | **Graph Theory Applications**
Date: May 2, 2013 | Spring 2013

Problem 1. (a) Since the 5-cycle is a subgraph of the Petersen graph, which needs at least three colors, $\chi(G) \geq 3$. Brook's theorem says that $\chi(G) \leq \Delta(G) = 3$. Therefore, $\chi(G) = 3$. A proper coloring with 3 colors is shown in the figure.
(b) Again, since the graph contains a 5-cycle, $\chi(G) \geq 3$. Brook's theorem says that $\chi(G) \leq \Delta(G) = 5$. A proper 4-coloring is shown in the figure. To prove that there is no proper coloring with 3 colors, notice that fixing a color for the central vertex forces the use of one or two new colors connected to it. This in turn forces the outer cycle to use at least two new or one new color, giving a total of four.



(a) Petersen Graph



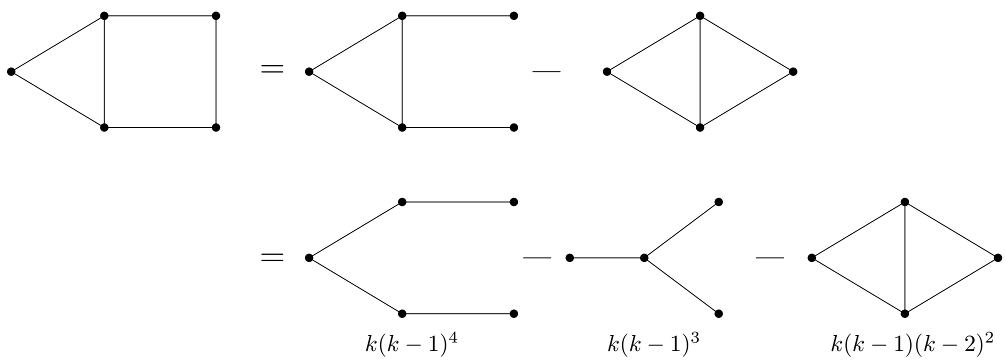
(b) Grötzsch Graph

Problem 2. Let $|E| = |V| + k$. Our goal is to show that G can be colored using at most $k+3$ colors. We note first that this quantity is always nonnegative, for since $G = (V, E)$ is a connected graph, G must have a spanning tree and consequently $k \geq -1$. Let T be any spanning tree of G. Since T is a tree, it is 2-colorable; this coloring may fail for G, though, because some of the $k+1$ extra edges which are present in G but not in T may be between nodes which are assigned the same color. These $k+1$ extra edges can be adjacent to at most $2(k+1)$ nodes; if we pick one end node from each of the $k+1$ edges and color each of the $k+1$ nodes using a distinct new color, we will obtain a proper coloring. The total number of colors used is $2 + (k+1) = k+3$.

Problem 3. Order the points based on their x-coordinate and start coloring them greedily. Every point that we have to color has at most two neighbors already colored (any half-plane can contain at most 2 neighbors), so three colors are sufficient.

Problem 4. The steps of the recursion are shown below. The first graph has chromatic polynomial $k(k-1)^4$ because the central vertex has k choices and all the others have $(k-1)$ choices. The second graph is similar. For the third graph, the left vertex has k choices, the top-middle has $(k-1)$ and the edges force the remaining to have $(k-2)$ choices each. So finally, the graph has the chromatic polynomial

$$\pi_k(G) = k(k-1)^4 - k(k-1)^3 - k(k-1)(k-2)^2 = k^5 - 6k^4 + 14k^3 - 15k^2 + 6k$$



Problem 5. We will use induction. Every tree T_n with $n \geq 2$ vertices has a vertex of degree 1. Let us call it u and the vertex that it is connected to v. By the recursion, we have

$$\pi_k(T_n) = \pi_k(T_n - (uv)) - \pi_k(T_n \cdot (uv))$$

But notice that $T_n - (uv)$ is a tree with $n-1$ vertices and an isolated vertex. The isolated vertex can be colored in k ways and the tree with $n-1$ vertices can be colored in $\pi_k(T_{n-1})$ ways. Also, $T_n \cdot (uv)$ is simply a tree with $n-1$ vertices. By the induction hypothesis

$$\pi_k(T_n) = k\pi_k(T_{n-1}) - \pi_k(T_{n-1}) = (k-1)\pi_k(T_{n-1}) = (k-1) \cdot k(k-1)^{n-2} = k(k-1)^{n-1}$$

The base case for $n = 1$ consists of an isolated vertex which can indeed be colored in k ways.

Problem 6. Using the recursive formula, we have for an edge e in C_n

$$\pi_k(C_n) = \pi_k(C_n - e) - \pi_k(C_n \cdot e) = \pi_k(P_n) - \pi_k(C_{n-1})$$

where P_n is the path graph on n nodes. But a path graph is also a tree, which means from the previous problem $\pi_k(P_n) = k(k-1)^{n-1}$. Now assume that the claim holds for the $(n-1)$-cycle. So for the n cycle we have

$$\pi_k(C_n) = k(k-1)^{n-1} - \pi_k(C_{n-1}) = k(k-1)^{n-1} - \big((k-1)^{n-1} + (-1)^{n-1}(k-1)\big) = (k-1)^n + (-1)^n(k-1)$$

The base case for $n = 3$ is easy to verify.