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Exercise 7 **Graph Theory Applications**
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Problem 1. If G is bipartite, every subgraph H of G is also bipartite. Now take the larger partition of H. It obviously forms an independent set of H which contains at least half the vertices of H. Conversely suppose that for every subgraph H of G we have $\alpha(H) \geq |V(H)|/2$. We want to show that G is a bipartite graph. If it is not, then G has an odd cycle H. Since $\alpha(H) < |V(H)|/2$, we have a contradiction.

Problem 2. Represent the conferences by vertices and join two vertices by an edge to represent the researcher who has attended those two conferences to construct graph G_1. Now construct graph G_2 which is a dual of G_1 by having one vertex for each researcher and an edge between researchers if they have attended a conference together. The statement of the problem asks for the maximum independent set of G_2. Notice that two researchers are strangers to each other if they do not share an edge in G_2 and consequently do not share a vertex in G_1. Therefore any maximum set of strangers in G_2 represents a maximum matching in G_1, which can be computed efficiently. (In fact duals of simple graphs are called *line graphs* and they have many interesting properties).

Problem 3. Suppose that G has m edges. Let x and y be two vertices in G which are joined by an edge. If $d(v)$ is the degree of a vertex v, for a triangle free graph $d(x) + d(y) \leq n$. This is because every vertex in the graph G is connected to at most one of x and y. Note now that

$$\sum_x d^2(x) = \sum_{xy \in E} (d(x) + d(y)) \leq mn$$

On the other hand, since $\sum_x d(x) = 2m$, by the Cauchy-Schwarz inequality

$$\sum_x d^2(x) \geq \frac{\left(\sum_x d(x)\right)^2}{n} \geq \frac{4m^2}{n}$$

Therefore, $m \leq n^2/4$. Hence, $m \leq \lfloor n^2/4 \rfloor$.

Problem 4. A graph G does not have an independent set of size 3 if and only if its complement $\bar{G}$ does not contain contain a clique of size 3. A graph G has at least m edges if and only if is complement G has at atmost $\binom{n}{2} - m$ edges. If $\bar{G}$ does not contain a triangle, the above proof shows that it must contain at most $\lfloor n^2/4 \rfloor$ edges. Therefore, $\binom{n}{2} - m \leq \lfloor n^2/4 \rfloor$ which implies that G must contain at least $\binom{n}{2} - \lfloor n^2/4 \rfloor$ edges, which is roughly $n^2/4$.

Problem 5. Label the vertices of the complete graph K_n by $1, 2, \ldots, n$ and color its edges by k colors, so that the color of the edge ab is equal to the number of the class (among the given k classes) that contains $|a - b|$, for all a and b. Picking $n = r(3, \ldots, 3)$, we can find a monochromatic triangle corresponding to one of the colors. Suppose the numbers corresponding to the triangle are a, b, c where $a < b < c$. Notice that the numbers $x = |a - b|$, $y = |b - c|$ and $z = |c - a|$ lie in the same class and $x + y = z$.