ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Exercise 7 | **Graph Theory Applications**
Date: April 25, 2013 | Spring 2013

Problem 1. Prove that a graph G is bipartite if and only if every subgraph H of G satisfies the inequality $\alpha(H) \geq |V(H)|/2$ in which $\alpha(H)$ is the size of a maximum independent set of H and $|V(H)|$ is the number of vertices of H.

Problem 2. Consider a group of researchers each of whom has attended exactly two conferences during a year. Two researchers know each other if and only if they have attended a conference together. Can you find a way to efficiently compute the largest subset of researchers who are complete strangers to each other? (Hint: There is no known efficient way to compute maximum independent sets for general graphs. There is something special here!)

Problem 3. *Turan's theorem* says that for every integer numbers $k < n$, the maximum number of edges of a graph G with n vertices and with no clique of size k is $(1 - \frac{1}{k-1})\frac{n^2}{2}$. Prove Turan's theorem for $k = 3$, i.e. show that the *maximum* number of edges of a graph with no triangles is $\lfloor \frac{n^2}{4} \rfloor$.

Problem 4. Using the above result, what can be said about the *minimum* number of edges in a graph containing no independent sets of size 3?

Problem 5. The concept of Ramsey numbers can be generalized to k classes in the following way. Assume coloring the edges of the complete graph K_n with k colors. Then, for sufficiently large n, every coloring of K_n must contain a monochromatic clique of color i of size α_i (for any one of the k colors). In other words, given $\alpha_1, \ldots, \alpha_k$, for a sufficiently large n, every coloring of K_n with k colors will have a monochromatic clique of size α_i for some color i. The least such number is denoted as $r(\alpha_1, \ldots, \alpha_k)$. Use this to prove the following.

The first n natural numbers are split into k classes. Assuming $r(\overbrace{3, 3 \ldots, 3}^{k \text{ times}})$ is finite, prove that if n is large enough (w.r.t. k), then one of the classes contains three integers x, y, z such that $x + y = z$.