ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Solution 6 | **Graph Theory Applications**
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Problem 1. (a) 2

(b) 2 if n is even, 3 otherwise.

(c) For $\chi'(W_2) = 1$, $\chi'(W_3) = 3$, and $\chi'(W_{n+1}) = n$ for $n \geq 3$.

Problem 2. We know that $\Delta \leq \chi' \leq \Delta + 1$. Assume $\chi' = \Delta$. This would mean that every colour is represented at every vertex. However, any set of edges of the same color gives a matching and hance covers even number of vertices. With odd number of vertices it is not possible that any color covers every vertex, so this contradicts $\chi' = \Delta$. We conclude that $\chi' = \Delta + 1$.

Problem 3. Consider coloring of the edges with using $q = \chi'$ colors $1, 2, \ldots, q$ and let E_i denote the set of edges with color i. Clearly, each of the E_i's defines a matching. Then

$$m = |E_1| + |E_2| + \ldots + |E_q| \leq qm^*$$

The required result follows.

Problem 4. Assume w.l.o.g. that $m \geq n$, and therefore $\Delta(K_{m,n}) = m$. Let $u_0, \ldots, u_{m-1}$ be the vertices on the left-hand side and $v_0, \ldots, v_{n-1}$ the vertices on the right-hand side. Also, let $c_0, \ldots, c_{m-1}$ be m distinct colors. We will suggest a coloring of $K_{m,n}$ with m colors and prove that it is a proper m-edge-coloring of this graph. In $K_{m,n}$, every vertex on the left-hand side is connected to every vertex on the right-hand side. Let $e_{i,j}$ be the edge connecting vertex u_i to vertex v_j, for all $0 \leq i \leq m-1$, $0 \leq j \leq n-1$. Then color edge $e_{i,j}$ by color $c_{(i+j) \mod m}$.

We now need show that this coloring is correct, i.e., no vertex has any incident edges colored by the same color. First consider a vertex u_i. The set of edges incident to u_i is $e_{i,0}, \ldots, e_{i,n-1}$ and these edges are assigned colors $c_{(i+0) \mod m}, \cdots c_{(i+n-1) \mod m}$. Since $n \leq m$, for $0 \leq x \leq n-1$, it holds that the $(i+x) \mod m$ correspond to n distinct elements of $0, \ldots, m-1$. Therefore our coloring assigned different colors to each of the n edges. On the other hand, consider a vertex v_j. The set of edges incident to v_j is $e_{0,j}, \ldots, e_{m-1,j}$ and is assigned colors $c_{(0+j) \mod m}, \ldots, c_{(m-1+j) \mod m}$. By the same argument, for $0 \leq x \leq m-1$, we get that the $(x+j) \mod m$ form a permutation of $0, \ldots, m-1$ (in fact, they correspond to a cyclic shift of the latter set j positions to the left) and therefore the colors assigned to the m edges are distinct. We conclude that our edge coloring is valid.

Problem 5. First note that G has even number of nodes, because $2|E| = 3|V|$. Take the union of two partitions corresponding to distinct colors c_1 and c_2 in te 3-colouring of G. In this subgraph every vertex has degree two (one edge for each color). Hence, this subgraph is a union of cycles. Further, since the subgraph is 2-colourable, every cycle has even edges. If the subgraph consists of a single even cycle, then it is a Hamiltonian. If there are more than one partitions in this subgraph, then in one of the partitions we exchange colors c_1 and c_2. The resulting colouring is proper with a different partitioning of the edges, which contradicts the uniqueness of the colouring. Hence the subgraph has only one partition, which is a Hamiltonian cycle.

Problem 6. Since G is 3-regular then it must have an even number of vertices. Suppose G is Hamiltonian, then any Hamiltonian cycle of G is even, so we can color its edges properly with 2 colors, say red and blue. Now each vertex is incident with 1 red edge, 1 blue edge and 1 uncolored edge. The uncolored edges form a matching of G, so we can color all of them with the same color, say green. Thus, G must be 3-edge-colorable, which is impossible. Therefore, G cannot be Hamiltonian.