ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Exercise 5 **Graph Theory Applications**
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Problem 1. The length of the maximum matching is exactly $\lfloor n/2 \rfloor$.

Problem 2. Use induction on n.

Problem 3. Suppose that $G(V;E)$ is a tree with two distinct perfect matchings, M_1 and M_2. Consider the graph $G' = (V; (M1 - M2) \cup (M2 - M1))$, i.e. G' is the subgraph of G containing those edges that occur in exactly one of the matchings. G' must have some edges because we are assuming that $M_1 \neq M_2$. Hence, G' has a connected component, C, which contains more than one vertex.

Now we will show that C contains a cycle, contradicting the assumption that G is a tree. Start at an arbitrary vertex $v_1 \in C$. Since C is connected and both matchings are perfect, M_1 contains an edge from v_1 to some $v_2 \in C$. Likewise, since M_2 is a perfect matching, it must contain an edge from v_2 to some $v_3 \in C$, and furthermore, by the definition of C, the edge $(v_2; v_3)$ is different from the edge $(v_1; v_2)$. Continuing in this way, alternating between edges from M_1 and M_2, we can continue to construct such a path for as long as we like. However, C is of finite size, and therefore, we must eventually find a cycle in C, contradicting the assumption that G was a tree.

Problem 4. Suppose that there is a perfect matching in G. This implies that $|V| = 2n$ for some n. Let M be a perfect matching. The game evolves in rounds, and at each round $k \geq 1$ first Player 1 picks a vertex and then Player 2. Let $v_1, v_2, v_3, v_4, \ldots$ be the sequence of moves, where odd indices correspond to moves of Player 1 and even indices to moves of Player 2, and the v_i's are some mapping over the vertices of G. The winning strategy of Player 2 is the following: at round k, Player 2 chooses as v_{2k} the vertex that is matched with v_{2k-1} in M. He can clearly choose this vertex: it is adjacent to v_{2k-1} and has not been visited yet, since by construction, the $2k-2$ vertices visited in the prior $k-1$ rounds are all pairs in M. Therefore Player 2 can always pick a vertex after Player 1, and therefore he has a winning strategy.

Conversely, suppose that Player 2 has a winning strategy. We need show that G has a perfect matching. We will do this by taking any non-perfect matching M and showing that there is an augmenting path for M. Then G must have a perfect matching. Consider any matching that is not perfect. Then there must be at least one vertex that is not covered by M: M only covers $2|M| < |V|$ vertices (since it is not perfect). Let v_1 be one of the vertices that are not covered by M. Suppose that Player 1 picks v_1 as his first move. Then we know that Player 2 has a move v_2 he can make (remember, he has a winning strategy!). Now if v_2 is not matched in M then we simply add $(v_1; v_2)$ to M; this is a trivial augmenting path. Otherwise, if v_2 is matched in M, Player 1 picks as v_3 the vertex that is matched with v_2 in M. Since Player 2 has a winning strategy, there exists some v_4 adjacent to v_3 and not yet visited that he can pick. If v_4 is in M then Player 1 can pick the vertex v_5 that is matched to v_4 in M; otherwise, we exhibited the augmenting path

$v_1, \ldots, v_5$. In other words, we continue in the same way until Player 2 (who can always pick a vertex after Player 1 since he has a winning strategy) picks a vertex not covered by M. Since M is finite, this eventually happens and with this we have found an augmenting path.