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Exercise 3 | **Graph Theory Applications**
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Problem 1. A permutation matrix is a matrix obtained by permuting the rows of the $n \times n$ identity matrix according to some permutation π of the numbers 1 to n, where

$$\pi = \{\pi(1), \pi(2), \ldots, \pi(n)\}$$

Every row and column therefore contains precisely a single 1 with 0s everywhere else, and every permutation corresponds to a unique permutation matrix.

For example, if matrix P is a 3×3 matrix defined by permutation

$$\pi = \{\pi(1) = 1, \pi(2) = 3, \pi(3) = 2\},$$

then PAP^T is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,3} & a_{1,2} \\ a_{3,1} & a_{3,3} & a_{3,2} \\ a_{2,1} & a_{2,3} & a_{2,2} \end{pmatrix}$$

It is easy to show (you should do it!) that left multiplying a matrix A by P results in permuting the rows of A according to π, i.e., row i of A becomes row $\pi(i)$ of PA, and right multiplying A by P^T results in permuting the columns of A according to π, i.e., column j of A becomes column $\pi(j)$ of AP^T . Therefore, the (i, j)-th entry of A, henceforth denoted as $A_{i,j}$, appears as the $(\pi(i), \pi(j))$-th entry of PAP^T .

Now suppose that $A_H = PA_GP^T$. Then we know that for all i, j,

$$A_{H_{i,j}} = (PA_GP^T)_{i,j} \implies A_{H_{\pi(i),\pi(j)}} = (PA_GP^T)_{\pi(i),\pi(j)} = A_{G_{i,j}}$$

where the last equality follows from the discussion above. We conclude that if vertices i and j of G are adjacent then vertices $\pi(i)$ and $\pi(j)$ of H are adjacent (since the corresponding entries of the adjacency matrices are the same). Then letting $\theta(i) = \pi_i$ map the vertices of G to the vertices of H we conclude that two vertices i and j are adjacent in G if and only if their images $\theta(i)$ and $\theta(j)$ are adjacent in H, and therefore G and H are isomorphic.

Problem 2. Let A be the incidence matrix of dimension $m \times n$ as in the hint. Our goal is to show that the rows of A are linearly independent over the binary field, which is equivalent to showing that the rank of A is m under binary operations. We claim that this will immediately imply that no more than n clubs can be formed under conditions 1 and 2. Indeed, since for any $m \times n$ matrix A it holds that $\text{rank}(A) \leq \min\{n, m\}$ showing that $\text{rank}(A) = m$ will imply that $n \geq m$, and therefore will complete the proof.

Consider the $m \times m$ matrix $AA^\top$. By the known inequality, $\text{rank}(AA^\top) \leq \text{rank}(A)$. So it suffices to show that $\text{rank}(AA^\top) = m$. It is now easy to check that rules 1 and 2 result in $AA^\top$ being the $m \times m$ identity matrix, hence $\text{rank}(AA^\top) = m$.

Problem 3. Remember that a tournament is a complete graph with an orientation assigned to each edge. The definition of a tournament ensures that $A + A^\top = J - I$, where J is the matrix with all 1 elements and I is the identity matrix. We prove that the rank of A is at least $n - 1$ by contradiction: Assume the rank of A is at most $n-2$; i.e., there are at least 2 linearly independent vectors x and y such that $xA = 0$ and $yA = 0$. Note furthermore that rank of J is 1; i.e. there is a non-zero linear combination $z = \alpha x + \beta y$ such that $zJ = 0$. Furthermore, $zA = 0$. We now compute

$$0 = z(A + A^\top)z^\top = z(J - I)z^\top = -zz^\top < 0,$$

which contradicts the existence of non-zero z.

Problem 4. We need to show that

$$\det(A - kI) = 0$$

We have,

$$\begin{aligned}
\det(A - kI) &= \det \begin{pmatrix} a_{1,1} - k & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - k & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - k \end{pmatrix} \\
&= \det \begin{pmatrix} (\sum_{j=1}^n a_{1,j}) - k & a_{1,2} & \dots & a_{1,n} \\ (\sum_{j=1}^n a_{2,j}) - k & a_{2,2} - k & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ (\sum_{j=1}^n a_{n,j}) - k & a_{n,2} & \dots & a_{n,n} - k \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} - k & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n,2} & \dots & a_{n,n} - k \end{pmatrix} = 0
\end{aligned}$$

where the second equality follows from properties of the determinant and the third from the fact that in a k-regular graph, every vertex i has exactly k neighbors, therefore each row (and each column) of A sum up to k.

Problem 5. For the "if" part, we want to show that $\det(A + kI) = 0$. If G is bipartite, then we can relabel the vertices such that A looks like

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix},$$

where B is a square matrix. (B is square from regularity.) So, $A + kI$ has the following structure:

$$A + kI = \left(\begin{array}{ccccc|ccccc}
k & 0 & & \dots & 0 & & & & & \\
0 & k & 0 & \dots & 0 & & & & & \\
\vdots & \ddots & \ddots & \ddots & \vdots & & & B & & \\
0 & \dots & 0 & k & 0 & & & & & \\
0 & & \dots & 0 & k & & & & & \\
\hline
 & & & & & k & 0 & & \dots & 0 \\
 & & & & & 0 & k & 0 & \dots & 0 \\
 & & B^\top & & & \vdots & \ddots & \ddots & \ddots & \vdots \\
 & & & & & 0 & \dots & 0 & k & 0 \\
 & & & & & 0 & & \dots & 0 & k
\end{array}\right).$$

Take the vector

$$v = (\underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}).$$

From the regular property it is clear that

$$v(A + kI) = 0,$$

so v is a non-trivial linear combination of the rows which results 0, thus $\det(A + kI) = 0$.

For the "only if" part, we know that there exists a v for which

$$v(A + kI) = 0. \tag{1}$$

We can also assume that the largest (absolute value) element of v is 1. (If not, we divide the vector by its largest element.) Let 1 be the i-th element of v. Consider the i-th column of $A' = A + kI$. We know that the diagonal elements of A' are at least k, so we have from (1) that

$$\sum_{j=1, j \neq i}^{n} v_j {A'}_{j,i} \leq -k. \tag{2}$$

Also, from regularity,

$$\sum_{j=1, j \neq i}^{n} {A'}_{j,i} \leq k. \tag{3}$$

Since v_i is the largest element in v, we must have that (2) and (3) are equalities and $v_j = -1$, if $A'_{j,i} \neq 0$. In other words, vertex i doesn't have self-loops and every vertex j that is connected to i has $v_j = -1$.

Consider a j for which (i, j) is an edge. We apply the same argument on the j-th column of A'. The same holds, only the sign switch,

$$\sum_{\ell=1, \ell \neq j}^{n} v_\ell {A'}_{\ell,j} \geq k,$$
$$\sum_{\ell=1, \ell \neq j}^{n} {A'}_{\ell,j} \leq k,$$

so $v_\ell = 1$ for every ℓ for which (j, ℓ) is an edge. We can go on with the same reasoning and since G is connected we eventually give a constraint on every element of v being equal to 1 or -1 and none of the vertices can have self-loops. Thus every edge has the property that it connects two vertices (i, j) with $v_i = -v_j$. Consequently, the sign of the elements in v gives a partitioning of the vertices, hence G is bipartite.

Problem 6. Let $R = BB^T$ and let d_i denote the i-the diagonal entry of matrix D. We need to show $r_{i,i} = d_i$ for $1 \leq i \leq n$, and $r_{i,j} = a_{i,j}$ for $1 \leq i,\ j \leq n$ and $i = j$. Let's start by looking at the diagonal entries of R. We have

$$r_{i,i} = \sum_{k=1}^{m} b_{i,k} b_{i,k} = \sum_{k=1}^{m} b_{i,k}^2$$

Since

$$b_{i,k} = 1, \text{ iff vertex } i \text{ neighbors edge } k$$

we have that $b_{i,k}^2 = 1$ iff vertex i neighbors edge k. Hence $\sum_{k=1}^{m} b_{i,k}^2$ counts the number of edges incident to vertex i, which is just d_i. Since $a_{i,i} = 0$ for $1 \leq i \leq n$ in simple graphs, we conclude

$$R_{i,i} = d + i = d_i + a_{i,i}$$

We now move to entries $R_{i,j}$, with $i \neq j$. We have

$$R_{i,j} = \sum_{k=1}^{m} b_{i,k} b_{k,j}$$

Again by definition

$$b_{i,k} = 1, \text{ iff vertex } i \text{ neighbors edge } k, \text{ and}$$
$$b_{k,j}^T = 1, \text{ iff edge } k \text{ neighbors vertex } j$$

Therefore $b_{i,k} b_{k,j}^T = 1$ iff edge k neighbors vertices i and j. Hence $\sum_{k=1}^{m} b_{i,k} b_{k,j}^T$ counts the number of edges that join vertices i and j, which by definition equals entry $a_{i,j}$ of the adjacency matrix (for simple graphs, this number can either be 0 or 1). The problem statement follows.