ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Exercise 3 **Graph Theory Applications**
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Problem 1. Let P be a permutation matrix. Show that if A_1 is the adjacency matrix of the graph, then $A_2 = P^\top A_1 P$ is an adjacency matrix corresponding to the same graph with the vertices renumbered.

Problem 2. In this exercise we will use some basic properties of rank of matrices. Given an $n \times m$ matrix A, $\text{rank}(A) \leq \min\{n, m\}$. Moreover, $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$, where A and B are matrices such that the number of columns of A equals the number of rows of B. All operations we consider are binary.

The n inhabitants of a town organize themselves into clubs subjsect to the following two conditions:

1. Each club must have an odd number of members.
2. Each pair of clubs must share an even number of members.

Show that no more than n clubs can be formed.

Hint: consider the incidence matrix A where rows correspond to clubs and columns to inhabitants. What is $AA^\top$ equal to?

Problem 3. The adjacency matrix of a directed graph D is the $n \times n$ matrix $A_D = (a_{u,v})$, where $a_{u,v}$ is the number of arcs with tail u and head v. Let A be the adjacency matrix of a tournament on n vertices. Show that $\text{rank}(A)$ is either n or $n - 1$.

Problem 4. Let G be a regular graph with degree k. Show that k is an eigenvalue of the adjacency matrix A of the graph G.

Problem 5. Continuing with the above problem, show that for a k-regular connected graph G, $-k$ is also an eigenvalue of A if and only if G is bipartite.

Problem 6. Show that for any graph G with incidence matrix B and adjacency matrix A,

$$BB^\top = A + D,$$

where D denotes a matrix whose diagonal element $(i; i)$ equals the degree of the vertex i, i.e. $d_{i,i} = d(i)$, while the off-diagonal elements $d_{i,j}$ are zero.