ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Solution 2 **Graph Theory Applications**

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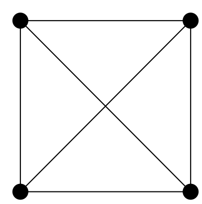
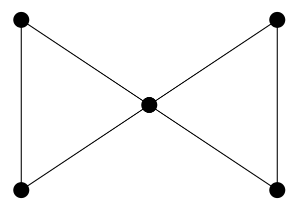
Problem 1. As shown in the figure, consider the complete graph on 4 vertices. This graph has a Hamiltonian cycle. However the degrees of its vertices are 3, 3, 3, 3, so it does not have an Eulerian



(i) K_4



(ii) A Bowtie graph

walk. For the second part, consider the bowtie graph shown above. It is easy to see that this graph has an Eulerian cycle but has no Hamiltonian cycle.

Problem 2. We first show that if G is Eulerian, then $d^+(v) = d^-(v)$ for every vertex $v \in V$. Consider an Eulerian circuit. If it visits a vertex k times, it needs both to arrive and leave this vertex, and thus the indegree and outdegree of each vertex is equal.

For the converse, assume that $d^+(v) = d^-(v)$ for every $v \in V$. Since G is nontrivial and weakly connected with all the indegrees and outdegrees equal, we can start from an arbitrary vertex and traverse edges to find a cycle in G, say $C_1 = (V_1, E_1)$. Note that this is always possible as if we get stuck somewhere, either the indegrees and outdegrees do not match or the graph is not connected. Remove from G all edges in E_1 and call the resulting graph G_1. If G_1 has no edges we are done. If G_1 still has edges, then G_1 still has equal indegree and outdegree on all vertices and thus we can repeat the previous argument and find another cycle C_2. We can continue this procedure until there are no more edges remaining. Now, let C_1 be one of the cycles in the partition. If G only consists of C_1, then G is obviously Eulerian. Otherwise, there exists another cycle C_2 with a common vertex v with C_1. We can then create a walk stitching together the cycles C_1 and C_2 and visiting all the edges in them. (You should verify that this is possible). By continuing this process we can create a closed circuit containing all edges of G. Thus G is Eulerian.

Problem 3. We will use induction on the number of vertices in the graph. Clearly the statement holds for $n = 2$. Assume the statement is true for a tournament with n vertices and consider a tournament on $n+1$ vertices. Let G' be the graph we get from G by taking out one of the vertices, say, v_{k+1} (and all of its adjacent edges). Clearly, G' is also a tournament and by the induction hypothesis, it has a directed Hamiltonian path, say $v_1 \to v_2 \to \ldots \to v_k$. Now look at G: if there is an edge directed from v_{k+1} to v_1 or from v_k to v_{k+1} we are done (just extend the path in G' to a path in G). Otherwise, there is an edge from v_1 to v_{k+1} and from v_{k+1} to v_k. There are three possibilities:

(a) all edges from v_{k+1} to v_i, $2 \leq i \leq k-1$ are directed from v_{k+1} to v_i; then the Hamilton path is $v_1 \rightarrow v_{k+1} \rightarrow v_2 \ldots v_k$;
(b) all edges from v_{k+1} to v_i, $2 \leq i \leq k-1$ are directed from v_i to v_{k+1}; then the Hamilton path is $v_1 \rightarrow v_2 \ldots v_{k-1} \rightarrow v_{k+1} \rightarrow v_k$;
(c) let $1 \leq i \leq k-1$ be the smallest index such that there is an edge directed from v_i to v_{k+1} and an edge directed from v_{k+1} to v_{i+1}. (There must be such an i otherwise we would be in case (a) or (b)). Then in the path in G', replace $(v_i \rightarrow v_{i+1})$ by $v_i \rightarrow v_{k+1} \rightarrow v_{i+1}$ to get a Hamiltonian path in G.

Problem 4. Let $L = \{v_1, v_2, \ldots, v_l\}$ be the longest path in G (in the given order). First we show that $l \geq k+1$. If $l \leq k$ then consider all the neighbors of the vertex v_l. By assumption, v_l is of degree k or larger. This means that v_l has a neighbor other than $v_1, v_2, \ldots, v_{l-1}$. But in this case we can extend the path by 1 by including this neighbor, contradicting the maximality of L. Now consider the vertex v_1. By assumption it is connected to at least k vertices. Since L is the longest path in G, all of the neighbors of v_1 belong to this path. Further, since v_1 has degree $\geq k$, one of its neighbors v_t has to be from the set $\{v_{k+1}, \ldots, v_l\}$. Then $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_t \rightarrow v_1$ forms a cycle of length $\geq k+1$.