ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Solution 1	**Graph Theory Applications**
Date: Feb 21, 2013	Spring 2013

Problem 1. Frame the problem as a graph with a vertex for each person and an edge between two people if they have shaken hands. Clearly, there are n vertices in the graph and the number of people vertex i has shaken hands with corresponds to the degree of the graph. We claim that, the number of different degrees in the graph can be at most $n-1$. If there is a degree 0 vertex, then there cannot be a degree $n-1$ vertex, which makes the number of degrees equal to $n-2$. On the other hand, if there is no degree 0 vertex, there are at most $n-1$ degrees $(1, \ldots, n-1)$. Since there are n people, by the pigeon-hole principle, there must be at least 2 people with the same degree, which proves the claim.

Problem 2. Consider a fixed color c_k, with $1 \leq k \leq 8$. Since we have 20 balls of c_k and they are all placed in the 6 jars, by the pigeonhole principle there exists at least one jar that contains at least two balls of color c_k. Clearly, this is true for all colors c_k, i.e.,

$$\text{For every } 1 \leq k \leq 8, \text{ there is a jar } j_k \text{ that contains at least 2 balls of color } c_k.$$

But there are only 6 jars, so by the pigeonhole principle there is a jar that appears at least twice in the set $\{j_1, j_2, ..., j_8\}$, and therefore contains at least two balls of two different colors.

Problem 3. (a) No. The sum of the degrees is odd.

(b) No. In a bipartite graph $(V_1 \cup V_2, E)$, the sum of degrees of nodes in V_1 is equal to the sum of degrees of nodes in V_2. Hence $\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v) = \frac{1}{2} \sum_{v \in V_1 \cup V_2} d(v) = \frac{56}{2} = 28$. However we cannot find an integer partition of 28 using one 5 and a few 3s and 6s.

(c) No. The node with degree 8 has to be adjacent to all other nodes in the graph and in particular to the two nodes with degree 1. Therefore, the node with degree 7 does not have enough neighbors.

(d) No. The sum of the degrees is 10, which means the number of edges is 5. On the other hand, a forest on 5 vertices can have at most 4 edges.

Problem 4. Choose a vertex x in $V(G)$ and an edge $xy \in E(G)$ and consider the sets S_i and their neighborhoods $N(S_i)$ where

$$S_0 = \{x, y\},\ S_1 = N(S_0)$$
$$S_{i+1} = N(S_i) \setminus (S_{i-1} \cup S_i) \text{ for } 1 \leq i \leq \operatorname{diam}(G) - 2$$

Clearly, by the definition of diameter, $V(G) = \cup_i S_i$ and since the maximum degree is $\Delta(G)$,

$$V(G) \leq 2\left(1 + (\Delta(G)-1) + (\Delta(G)-1)^2 + \ldots + (\Delta(G)-1)^{\operatorname{diam}(G)-1}\right) = 2\frac{(\Delta(G)-1)^{\operatorname{diam(G)}} - 1}{\Delta(G) - 2}$$

Problem 5. The first inequality follows directly from the definitions:

$$\mathrm{rad}(G) = \min_{x \in V} ecc(x) \text{ and } \mathrm{diam}(G) = \max_{x \in V} ecc(x)$$

Next, let $x^* \in V$ be the (or one of the) most "central" vertices in G, i.e., such that $ecc(x^*) = rad(G)$. By definition, $ecc(x^*) = \max_{u \in V} d(x*, u)$. Therefore, for *every* vertex $u \in V$

$$d(x^*, u) \leq ecc(x^*)$$

Now for every pair of vertices (u, v), the shortest path between u and v cannot be longer than the path that goes from u to v through x^*. Therefore,

$$d(u, v) \leq d(u, x^*) + d(x^*, v)$$

Combining the above two

$$d(u, v) \leq d(u, x^*) + d(x^*, v) \leq 2ecc(x^*)$$

Let (a, b) be the (or one of the) pair of vertices such that $\mathrm{diam}(G) = d(a, b)$. Since the inequality above holds for every pair of vertices it also holds for (a, b), which proves the second inequality of the problem.

For $G = K_3$, $\mathrm{rad}(G) = \mathrm{diam}(G)$, and if G is a star, then $2\mathrm{rad}(G) = \mathrm{diam}(G)$.