ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 3 — Information Theory and Coding

Solutions to homework 1 — Sep. 27, 2011

PROBLEM 1. Note that $E_0 = E_1 \cup E_2 \cup E_3$.

(a) (1) For disjoint events, $P(E_0) = P(E_1) + P(E_2) + P(E_3)$, so $P(E_0) = 3/4$.

(2) For independent events, $1 - P(E_0)$ is the probability that none of the events occur, which is the product of the probabilities that each one doesn't occur. Thus $1 - P(E_0) = (3/4)^3$ and $P(E_0) = 37/64$.

(3) If $E_1 = E_2 = E_3$, then $E_0 = E_1$ and $P(E_0) = 1/4$.

(b) (1) From the Venn diagram in Fig. 1, $P(E_0)$ is clearly maximized when the events are disjoint, so $\max P(E_0) = 3/4$.

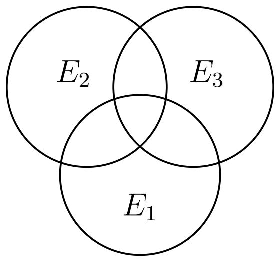

Figure 1: Venn Diagram for problem 1 (b)(1)

(2) The intersection of each pair of sets has probability 1/16. As seen in Fig. 2, $P(E_0)$ is maximized if all these pairwise intersections are identical, in which case $P(E_0) = 3(1/4 - 1/16) + 1/16 = 5/8$. One can also use the formula $P(E_0) = P(E_1)+P(E_2)+P(E_3)-P(E_1\cap E_2)-P(E_1\cap E_3)-P(E_2\cap E_3)+P(E_1\cap E_2\cap E_3)$, and notice that all the terms except the last is fixed by the problem, and the last term cannot be made more than $\min_{i,j} P(E_i \cap E_j) = 1/16$.

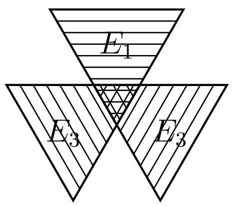

Figure 2: Venn Diagram for problem 1 (b)(2)

PROBLEM 2. Let L be the event that the loaded die is picked and H the event that the honest die is picked. Let A_i be the event that i is turned up on the first roll, and B_i be the event that i is turned up on the second roll. We are given that $P(L) = 1/3, P(H) = 2/3$; $P(A_1 \mid L) = 2/3$; $P(A_i \mid L) = 1/15 \quad 2 \le i \le 6$; $P(A_i \mid H) = 1/6 \quad 1 \le i \le 6$. Then

$$P(L \mid A_1) = \frac{P(L, A_1)}{P(A_1)} = \frac{P(A_1 \mid L)\,P(L)}{P(A_1 \mid L)\,P(L) + P(A_1 \mid H)\,P(H)} = \frac{2}{3}.$$

This is the probability that the loaded die was picked conditional on the first roll showing a 1. For two rolls we make the assumption from the physical mechanism involved in rolling a die that the outcome on the two successive rolls of a given die are independent. Thus $P(A_1B_1 \mid L) = (2/3)^2$ and $P(A_1B_1 \mid H) = (1/6)^2$. It follows as before that

$$P(L \mid A_1B_1) = \frac{8}{9}.$$

Problem 3. Since A, B, C, D form a Markov chain their probability distribution is given by

$$p(a)p(b|a)p(c|b)p(d|c) \tag{1}$$

(a) Yes: Summing (1) over d shows that A, B, C have the probability distribution $p(a)p(b|a)p(c|b)$.

(b) Yes: The reverse of a Markov chain is also a Markov chain. Applying this to A, B, C, D and using part (a) we get that D, C, B is a Markov chain. Reversing again we get the desired result.

(c) Yes: Since A, B, C, D is a Markov chain, given C, D is independent of B, and thus $p(d|c) = p(d|(b,c))$. So (1) can be written as

$$p(a,(b,c),d) = p(a)p((b,c)|a)p(d|(b,c)).$$

(d) Yes, by a similar (in fact easier) reasoning as (c).

Problem 4. No. Take for example $A = D$ and let A be independent of the pair (B, C). Then both A, B, C and B, C, A (same as B, C, D) are Markov chains. But A, B, C, D is not: A is not independent of D when B and C are given.

Problem 5.

(a) Note that the event $N = n$ is the same as the coin falling tails $n-1$ times followed by it falling heads. Since the coin flips are independent and they are fair, we get $\Pr(N = n) = 2^{-(n-1)}2^{-1} = 2^{-n}$. Using Bayes' rule:

$$\Pr(N = n|N \in \{n, n+1\}) = \frac{\Pr(N = n)}{\Pr(N \in \{n, n+1\})} = \frac{2^{-n}}{2^{-n} + 2^{-(n+1)}} = 2/3$$

(b) The only way we find 1 franc in the chosen box is when $N = 1$ and we have chosen the box with the smaller amount of money. The other box thus contains 3 francs.

(c) If we find 3^n francs in the chosen box, we know that N is either n (and the other box contains 3^{n-1} francs) or $n+1$ (and the other box contains 3^{n+1} francs). Using part (a), $N = n$ with probabity 2/3, and $N = n+1$ with probability 1/3. Thus the expected money in the other box is

$$\tfrac{2}{3}3^{n-1} + \tfrac{1}{3}3^{n+1} = \tfrac{11}{9}3^n \quad \text{francs.}$$

(d) Indeed, no matter what we find in the chosen box, the expected amount in the other box is more then the amount found in the chosen box (3 vs 1 as in part (b) or 11/9 times as in part (c)). We thus have, with X and Y representing the amount in the two boxes,

$$E[X|Y] > Y \qquad \text{and} \qquad E[Y|X] > X.$$

This appears to be a paradox if we take expectations again to obtain

$$E[X] > E[Y] \qquad \text{and} \qquad E[Y] > E[X].$$

However, some thought reveals that $E[X]$ and $E[Y]$ do not exist, and so the last equation is without content: Since $\Pr(N = n) = 2^{-n}$, the expected amount of money in the box with the smaller amount is $\sum_{n\geq 1} 2^{-n}3^{n-1}$ which is a divergent series.

Problem 6.

(a)

$$\begin{aligned}
E[X+Y] &= \sum_{x,y}(x+y)P_{XY}(x,y) \\
&= \sum_{x,y} xP_{XY}(x,y) + \sum_{x,y} yP_{XY}(x,y) \\
&= \sum_{x} xP_X(x) + \sum_{y} yP_Y(y) \\
&= E[X] + E[Y].
\end{aligned}$$

Note that independence is not necessary here and that the argument extends to non-discrete variables if the expectation exists.

(b)

$$\begin{aligned}
E[XY] &= \sum_{x,y} xyP_{XY}(x,y) \\
&= \sum_{x,y} xyP_X(x)P_Y(y) \\
&= \sum_{x} xP_X(x) \sum_{y} yP_Y(y) \\
&= E[X]\,E[Y].
\end{aligned}$$

Note that the statistical independence was used on the second line. Let X and Y take on only the values ± 1 and 0. An example of uncorrelated but dependent variables is

$$P_{XY}(1,0) = P_{XY}(0,1) = P_{XY}(-1,0) = P_{XY}(0,-1) = \frac{1}{4}.$$

An example of correlated and dependent variables is

$$P_{XY}(1,1) = P_{XY}(-1,-1) = \frac{1}{2}.$$

(c) Using (a), we have

$$\begin{aligned}
\sigma^2_{X+Y} &= E\big[(X - E[X] + Y - E[Y])^2\big] \\
&= E[(X - E[X])^2] + 2E[(X - E[X])(Y - E[Y])] + E[(Y - E[Y])^2].
\end{aligned}$$

The middle term, from (a), is $2(E[XY] - E[X]E[Y])$. For uncorrelated variables that is zero, leaving us with $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$.

PROBLEM 7. We solve the problem for a general vehicle with n wheels.

(a) Out of $n!$ possible orderings $(n-1)!$ has the tyre 1 in its original place. Thus tyre 1 is installed in its original position with probability $1/n$.

(b) All tyres end up in their original position in only 1 of the $n!$ orders. Thus the probability of this event is $1/n!$.

(c) Let X_i be the indicator random variable that tyre i is installed in its original position, so that the number of tyres installed in their original positions is $N = \sum_{i=1}^{n} X_i$. By (a), $E[X_i] = 1/n$. By the linearity of expectation, $E[N] = n(1/n) = 1$. Note that the linearity of the expectation holds even if the X_i's are not independent (as it is in this case).

(e) Let A_i be the event that the ith tyre remains in its original position. Then, the event we are interested in is the complement of the event $\bigcup_i A_i$ and thus has probability $1 - \Pr(\bigcup_i A_i)$. Furthermore, by the inclusion/exclusion formula,

$$\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i) - \sum_{i_1<i_2} \Pr(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \dots .$$

The jth sum above consists of $\binom{n}{j}$ terms, each term having the value $P(A_1 \cap \dots \cap A_j)$. Note that this is the probability of the event that tyres 1 through j have remained in their original positions, and equals $(n-j)!/n!$. Consequently,

$$\Pr\left(\bigcup_i A_i\right) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{(n-j)!}{n!} = \sum_{j=1}^{n} (-1)^{j-1} 1/j!,$$

and the event that no tyre remains in its original position has probability

$$1 - \Pr\left(\bigcup_i A_i\right) = \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$

(For the case $n = 4$, the value is 3/8.)