Solution to Test
Traitement Quantique de l'Information

Exercice 1

(a) Initial state is $|h\rangle$. After the first semi transparent mirror

$$H|h\rangle = \frac{1}{\sqrt{2}}(|h\rangle + |v\rangle).$$

After two mirrors

$$XH|h\rangle = \frac{1}{\sqrt{2}}(X|h\rangle + X|v\rangle) = \frac{1}{\sqrt{2}}(|v\rangle + |h\rangle).$$

After the second semi transparent mirror

$$\begin{aligned} HXH|h\rangle &= \frac{1}{\sqrt{2}}(H|v\rangle + H|h\rangle) \\ &= \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}(|h\rangle - |v\rangle) + \frac{1}{\sqrt{2}}(|h\rangle + |v\rangle)\right\} = |h\rangle . \end{aligned}$$

So just before the detectors the state is $|h\rangle$ thus

$$\text{Prob}(\mathcal{D}_1) = |\langle h|\, h\rangle|^2 = 1,\ \text{Prob}(\mathcal{D}_2) = |\langle v|\, h\rangle|^2 = 0.$$

(b) The condition for absorption of the photon by the atom is that the frequency of the photon be (see Einstein relation and Bohr model)

$$\nu = \frac{1}{h}(E_1 - E_0),$$

and that initially the electron be on the lowest energy level E_0.

(c) **Calculation for $|h\rangle$:**

$$S|h\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|h\rangle + |v\rangle),$$

$$R|h\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} = |v\rangle ,$$

$$P|h\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} = |abs\rangle .$$

Calculation for $|v\rangle$:

$$S|v\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|h\rangle - |v\rangle),$$

$$R|v\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = |h\rangle\,,$$

$$P|v\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} = |v\rangle\,.$$

Calculation for $|abs\rangle$:

$$S|abs\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} = |abs\rangle\,,$$

$$R|abs\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} = |abs\rangle\,,$$

$$P|abs\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = |h\rangle\,.$$

(d) Initially the state is $|h\rangle$. After the first semi transparent mirror :

$$S|h\rangle = \frac{1}{\sqrt{2}}(|h\rangle + |v\rangle).$$

After the two mirrors

$$RS|h\rangle = \frac{1}{\sqrt{2}}(|v\rangle + |h\rangle),$$

and after the "atom" :

$$PRS|h\rangle = \frac{1}{\sqrt{2}}(P|v\rangle + P|h\rangle) = \frac{1}{\sqrt{2}}(|v\rangle + |abs\rangle).$$

After the second semi transparent mirror :

$$SPRS|h\rangle = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|h\rangle + |v\rangle) + |abs\rangle) = \frac{1}{2}|h\rangle + \frac{1}{2}|v\rangle + \frac{1}{\sqrt{2}}|abs\rangle\,.$$

Therefore, the state of the photon just after the second semi transparent mirror (just before the detection) is $\frac{1}{2}|h\rangle + \frac{1}{2}|v\rangle + \frac{1}{\sqrt{2}}|abs\rangle$ and we have

$$\text{Prob}(\mathcal{D}_1) = \frac{1}{4},\ \text{Prob}(\mathcal{D}_2) = \frac{1}{4},\ \text{Prob(abs)} = \frac{1}{2},$$

because for example

$$\text{Prob(abs)} = |\langle abs|(\frac{1}{2}|h\rangle + \frac{1}{2}|v\rangle + \frac{1}{\sqrt{2}}|abs\rangle)|^2 = \frac{1}{2}.$$

(e) The matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not unitary. This can be checked explicitly. For example

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I.$$

Thus, it can not model the absorption process of the photon by the atom. The matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is unitary. Indeed

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus, it may model the absorption of the photon. Note also that this matrix acts like a Hadamard matrix on the subspace $\{|h\rangle, |abs\rangle\}$.

Exercice 2

(a) For an observable the matrix must be hermitian because the measurements (eigen-values) must be real numbers. Thus $A = A^\dagger = A^{*,T}$ is correct. The other expression $AA^\dagger = I$ means that A is unitary. Such matrices can have complex eigen-values and are not always observable.

(b) The orthonormal basis that models the measurement apparatus is $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$. The possible outcomes of the measurement are eigen-values α_j for A and eigen-state $|\phi_j\rangle$ for the state. The associated probabilities are $\text{Prob}(j) = |\langle\phi_j|\,\psi\rangle|^2$.

(c) The expected value of A is

$$\begin{aligned} \text{Exp}(A) &= \sum_{j=1}^{n} \alpha_j \text{Prob}(j) = \sum_{j=1}^{n} \alpha_j \langle\phi_j|\,\psi\rangle \overline{\langle\phi_j|\,\psi\rangle} \\ &= \sum_{j=1}^{n} \alpha_j \langle\psi|\,\phi_j\rangle \langle\phi_j|\,\psi\rangle \\ &= \langle\psi| (\sum_{j=1}^{n} \alpha_j |\phi_j\rangle \langle\phi_j|) |\psi\rangle . \end{aligned}$$

Using the spectral decomposition of A :

$$A = \sum_{j=1}^{n} \alpha_j |\phi_j\rangle \langle\phi_j| ,$$

we find that $\text{Exp}(A) = \langle\psi| A |\psi\rangle$.

(d) For the variance we have

$$\text{Var}(A) = \sum_{j=1}^{n} \alpha_j^2 \text{Prob}(j) - (\sum_{j=1}^{n} \alpha_j \text{Prob}(j))^2.$$

The second term on the r.h.s. is $(\langle\psi| A |\psi\rangle)^2$ by the previous calculation. For the first term we do the same calculation as before

$$\sum_{j=1}^{n} \alpha_j^2 \text{Prob}(j) = \langle\psi| (\sum_{j=1}^{n} \alpha_j^2 |\phi_j\rangle \langle\phi_j|) |\psi\rangle .$$

Since α_j^2 are the eigen-values of A^2 and $|\phi_j\rangle$ are still the eigen-vectors of A^2, we have

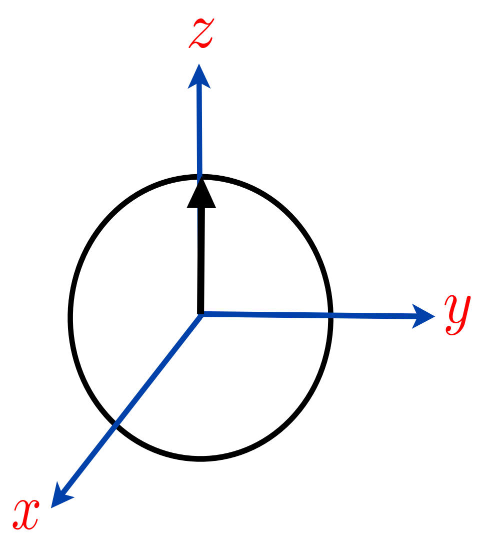
$$A^2 = \sum_{j=1}^{n} \alpha_j^2 |\phi_j\rangle \langle\phi_j| .$$

Thus, we find that $\sum_{j=1}^{n} \alpha_j^2 \text{Prob}(j) = \langle\psi| A^2 |\psi\rangle$. Putting things together, we find that

$$\text{Var}(A) = \langle\psi| A^2 |\psi\rangle - (\langle\psi| A |\psi\rangle)^2.$$

Exercice 3

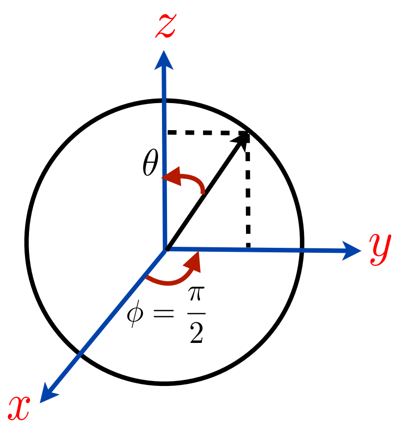
(a) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, thus from the formula gievn for the exponential of Pauli matrices $e^{i\theta\sigma_z} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. So $e^{i\theta\sigma_z} |\uparrow\rangle = e^{i\theta} |\uparrow\rangle$ and both vectors $|\uparrow\rangle$ and $e^{i\theta\sigma_z} |\uparrow\rangle$ are along z-axis.



(b) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $e^{i\theta\sigma_x} = I\cos(\theta) + i\sin(\theta)\sigma_x = \begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}$. Therefore, we have

$$e^{i\theta\sigma_x}\left|\uparrow\right\rangle = \begin{pmatrix} \cos(\theta) \\ i\sin(\theta) \end{pmatrix} = \cos(\theta)\left|\uparrow\right\rangle + i\sin(\theta)\left|\downarrow\right\rangle$$
$$\cos(\frac{2\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{2\theta}{2})\left|\downarrow\right\rangle$$

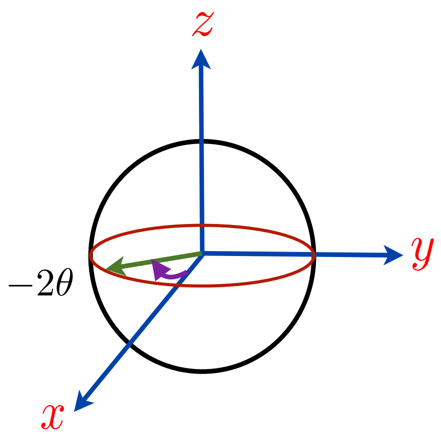
Note that the angle on the next figure is not θ but 2θ.



(c) Here $e^{i\theta\sigma_z} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ and $\frac{\left|\uparrow\right\rangle + \left|\downarrow\right\rangle}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Then

$$e^{i\theta\sigma_z}(\frac{\left|\uparrow\right\rangle + \left|\downarrow\right\rangle}{\sqrt{2}}) = \frac{e^{i\theta}\left|\uparrow\right\rangle + e^{-i\theta}\left|\downarrow\right\rangle}{\sqrt{2}} = e^{i\theta}\left\{\frac{1}{\sqrt{2}}\left|\uparrow\right\rangle + \frac{e^{-2i\theta}}{\sqrt{2}}\left|\downarrow\right\rangle\right\}.$$

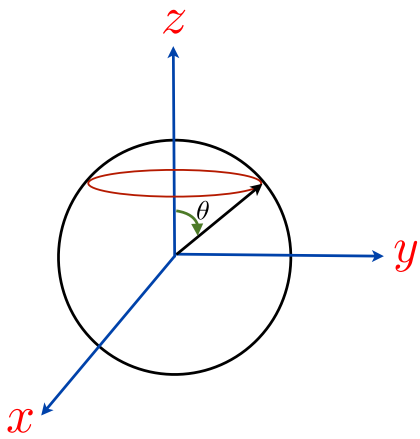
We have a vector with $(\theta, \phi) = (\frac{\pi}{2}, -2\theta)$.



(d) We have

$$e^{ita\sigma_z}(\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\phi}\sin(\frac{\theta}{2})|\downarrow\rangle) = \cos(\frac{\theta}{2})e^{ita}|\uparrow\rangle + e^{i\phi}\sin(\frac{\theta}{2})e^{-ita}|\downarrow\rangle$$
$$= e^{ita}\left\{\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i(\phi-2ta)}\sin(\frac{\theta}{2})|\downarrow\rangle\right\}.$$

The angle θ stays fixed and the angle ϕ varies as $\phi \to \phi - 2ta$. The resulting trajectory is a circle around the z-axis where the period is given by $-2Ta = -2\pi$ which gives $T = \frac{\pi}{a}$.



Exercice 4

(a) La banque ne veut pas détruire le billet quantique. Donc elle doit faire des mesures dans des bases qui ne perturbe pas les états des photons. A partir du numéro de série S elle sait que le photon numero i possède une p_i (égale à 0, 1, + ou −). Ainsi elle sait que le photon i a été préparé dans la base $\{|0\rangle, |1\rangle\}$ (base Z) ou bien dans la base $\{|+\rangle, |-\rangle\}$ (base X). Elle fait donc une mesure de la polarisation dans la même base que la base de préparation de l'état : ainsi l'étatsqui est déja un des vecteurs de base ne change pas (ou est projeté sur lui même avec probabilité 1 !). La banque observe ainsi que tous les photons sont dans un état correct, et ceci sans détruire le billet.

(b) Le malfaiteur ne peut pas copier le billet avec une seule machine quantique (unitaire) à cause du théorème de non-clonage (no-cloning theorem).

(c) Imaginons que le photon i a été initialement préparer dans létat $|0\rangle$ par la banque. Soit banque$|0\rangle$ cet évènement. Le malfaiteur lui, choisit un état aléatoire parmi $|0\rangle, |1\rangle, |+\rangle, |-\rangle$. Soit malf$|0\rangle$, malf$|1\rangle$, malf$|+\rangle$, malf$|-\rangle$ les évènements correspondants.
La banque va faire la mesure dans la base $\{|0\rangle, |1\rangle\}$ car elle sait que l'état attendu devrait être $|0\rangle$ (et elle ne veut pas détruire l'état a priori). Calculons la probabilité qu'elle détecte une erreur Prob(E|banque$|0\rangle$) quand elle a prépareé son photon initialement dans l'état $|0\rangle$.
En conditionnant sur les choix du malfaiteur on a

$$\begin{aligned}\text{Prob}(E \mid \text{banque}|0\rangle) =& \text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|0\rangle)\text{Prob}(\text{malf}|0\rangle)\\ &+ \text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|1\rangle)\text{Prob}(\text{malf}|1\rangle)\\ &+ \text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|+\rangle)\text{Prob}(\text{malf}|+\rangle)\\ &+ \text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|-\rangle)\text{Prob}(\text{malf}|-\rangle)\end{aligned}$$

Pour les probabilités conditionnelles on utilise la règle de Born en se rappelant que la banque utilise la base $\{|0\rangle, |1\rangle\}$ (puisque on conditionne sur l'évènement banque$|0\rangle$). La banque concluera á une erreur si elle observe $|1\rangle$:

$$\text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|0\rangle) = |\langle 1|0\rangle|^2 = 0$$

$$\text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|1\rangle) = |\langle 1|1\rangle|^2 = 1$$

$$\text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|+\rangle) = |\langle 1|+\rangle|^2 = \frac{1}{2}$$

$$\text{Prob}(E \mid \text{banque}|0\rangle; \text{malf}|-\rangle) = |\langle 1|-\rangle|^2 = \frac{1}{2}$$

Pour les probabilités des choix du malfaiteur on a 1/4 uniformément (malfaiteur qui se simplifie la vie !). Et donc

$$\text{Prob}(E \mid \text{banque}|0\rangle) = \frac{1}{2}$$

Finalement on note que ce calcul est le même quel que soit l'état initial préparé par la banque. En conditionnant par rapport à cet état initial,

$$\begin{aligned}\text{Prob}(E) = & \text{Prob}(E \mid \text{banque}|0\rangle)\text{Prob}(\text{banque}|0\rangle)\\ &+ \text{Prob}(E \mid \text{banque}|1\rangle)\text{Prob}(\text{banque}|1\rangle)\\ &+ \text{Prob}(E \mid \text{banque}|+\rangle)\text{Prob}(\text{banque}|+\rangle)\\ &+ \text{Prob}(E \mid \text{banque}|-\rangle)\text{Prob}(\text{banque}|-\rangle)\end{aligned}$$

Comme il y a 4 états initiaux uniformément répartis on trouve,

$$\text{Prob}(E) = 4 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}$$

Résumons : la probabilité qu'un photon soit observé dans un état incorrect par la baque est 1/2. En moyenne la banque trouve $N/2$ erreurs. En fait il y a en moyenne $N/4$ photons corrects et $3N/4$ photons incorrects. Les $N/2$ erreurs sont détectée parmi ces $3N/4$ photons incorrects.

Exercice 5

(a) Suppose that we can find two states with

$$|B_\theta\rangle = (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle).$$

Then, we have $\cos(\theta) = a_1a_2$ and $\sin(\theta) = b_1b_2$ and $a_1b_2 = 0$ and $b_1a_2 = 0$. Since $a_1b_2 = 0$ then $a_1 = 0$ or $b_2 = 0$. If $a_1 = 0$, we have $\cos(\theta) = 0 \to \theta = \frac{\pi}{2}$. If $b_2 = 0$ then $\sin(\theta) = 0 \to \theta = 0$. Since $b_1a_2 = 0$ then $b_1 = 0$ or $a_2 = 0$. We arrive at the same conclusion. Thus only when $\theta = 0$ or $\theta = \frac{\pi}{2}$ can we write $|B_\theta\rangle$ as a product state.

(b) The four operations of Alice are

$$\begin{aligned} I_1 \otimes I_2 \left| B_\theta \right\rangle &= \left| B_\theta \right\rangle = \cos(\theta) \left| 00 \right\rangle + \sin(\theta) \left| 11 \right\rangle, \\ \sigma_x^{(1)} \otimes I_2 \left| B_\theta \right\rangle &= \cos(\theta) \left| 10 \right\rangle + \sin(\theta) \left| 01 \right\rangle, \\ \sigma_z^{(1)} \otimes I_2 \left| B_\theta \right\rangle &= \cos(\theta) \left| 00 \right\rangle - \sin(\theta) \left| 11 \right\rangle. \\ i\sigma_y^{(1)} \otimes I_2 \left| B_\theta \right\rangle &= \cos(\theta) \left| 10 \right\rangle - \sin(\theta) \left| 01 \right\rangle. \end{aligned}$$

In the last equality, we use $i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Alice wants to send 00. Bob receives $\left| B_\theta \right\rangle$ because she just sends her photon to Bob. The measurement outcome of Bob are $\left| B_{00} \right\rangle, \left| B_{01} \right\rangle, \left| B_{01} \right\rangle, \left| B_{11} \right\rangle$, therefore

$$\begin{aligned} \text{Prob}(00) &= \left| \left\langle B_{00} \right| B_\theta \rangle \right|^2 \\ &= \frac{1}{2} |(\langle 00| + \langle 11|)(\cos(\theta) \left| 00 \right\rangle + \sin(\theta) \left| 11 \right\rangle)|^2 \\ &= \frac{1}{2}(\cos(\theta) + \sin(\theta))^2 \\ &= \frac{1}{2}(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta) \\ &= \frac{1+\sin(2\theta)}{2}. \end{aligned}$$

$$\begin{aligned} \text{Prob}(01) &= \left| \left\langle B_{01} \right| B_\theta \rangle \right|^2 \\ &= \frac{1}{2} |(\langle 01| + \langle 10|)(\cos(\theta) \left| 00 \right\rangle + \sin(\theta) \left| 11 \right\rangle)|^2 = 0. \end{aligned}$$

$$\begin{aligned} \text{Prob}(10) &= \left| \left\langle B_{10} \right| B_\theta \rangle \right|^2 \\ &= \frac{1}{2} |(\langle 00| - \langle 11|)(\cos(\theta) \left| 00 \right\rangle + \sin(\theta) \left| 11 \right\rangle)|^2 \\ &= \frac{1}{2}(\cos(\theta) - \sin(\theta))^2 \\ &= \frac{1}{2}(\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta) \\ &= \frac{1-\sin(2\theta)}{2}. \end{aligned}$$

and $\text{Prob}(11) = \left| \left\langle B_{11} \right| B_\theta \rangle \right|^2 = 0$. The messages observed by Bob are 00 or 10 with probabilities $\frac{1+\sin(2\theta)}{2}$ and $\frac{1-\sin(2\theta)}{2}$. Thus, the probability of error is $\text{Prob}(10) = \frac{1-\sin(2\theta)}{2}$. It is minimal if $\sin(2\theta) = 1 \to \theta = \frac{\pi}{4}$. So if $\left| B_{\theta=\frac{\pi}{4}} \right\rangle$ is a Bell state $\left| B_{00} \right\rangle$.
It is maximal if $\sin(2\theta)$ is minimal in $\theta \in [0, \frac{\pi}{2}]$. That is the case if $\theta = 0$, i.e., $\text{Prob}(10) = \frac{1}{2}$ and $\left| B_{\theta=0} \right\rangle = \left| 00 \right\rangle = \left| 0 \right\rangle \otimes \left| 0 \right\rangle$ a product state.