Solution de la série 7
Traitement Quantique de l'Information

Exercice 1 *Variation sur l'interféromètre de Mach-Zehnder.*

(a) The condition for absorption of the photon by the atom is that the frequency of the photon be (see Einstein relation and Bohr model)

$$\nu = \frac{1}{h}(E_1 - E_0),$$

and that initially the electron be on the lowest energy level E_0.

(b) **Calculation for $|h\rangle$:**

$$S\,|h\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|h\rangle + |v\rangle),$$

$$R\,|h\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} = |v\rangle\,,$$

$$P\,|h\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} = |abs\rangle\,.$$

Calculation for $|v\rangle$:

$$S\,|v\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|h\rangle - |v\rangle),$$

$$R\,|v\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = |h\rangle\,,$$

$$P\,|v\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} = |v\rangle\,.$$

Calculation for $|abs\rangle$:

$$S\,|abs\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} = |abs\rangle\,,$$

$$R\,|abs\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} = |abs\rangle\,,$$

$$P\,|abs\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = |h\rangle\,.$$

(c) Initially the state is $|h\rangle$. After the first semi transparent mirror :

$$S\,|h\rangle = \frac{1}{\sqrt{2}}(|h\rangle + |v\rangle).$$

After the two mirrors

$$RS\,|h\rangle = \frac{1}{\sqrt{2}}(|v\rangle + |h\rangle),$$

and after the "atom" :

$$PRS\,|h\rangle = \frac{1}{\sqrt{2}}(P\,|v\rangle + P\,|h\rangle) = \frac{1}{\sqrt{2}}(|v\rangle + |abs\rangle).$$

After the second semi transparent mirror :

$$SPRS\,|h\rangle = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|h\rangle + |v\rangle) + |abs\rangle) = \frac{1}{2}\,|h\rangle + \frac{1}{2}\,|v\rangle + \frac{1}{\sqrt{2}}\,|abs\rangle\,.$$

Therefore, the state of the photon just after the second semi transparent mirror (just before the detection) is $\frac{1}{2}\,|h\rangle + \frac{1}{2}\,|v\rangle + \frac{1}{\sqrt{2}}\,|abs\rangle$ and we have

$$\text{Prob}(\mathcal{D}_1) = \frac{1}{4},\ \text{Prob}(\mathcal{D}_2) = \frac{1}{4},\ \text{Prob(abs)} = \frac{1}{2},$$

because for example

$$\text{Prob(abs)} = |\,\langle abs|\,(\frac{1}{2}\,|h\rangle + \frac{1}{2}\,|v\rangle + \frac{1}{\sqrt{2}}\,|abs\rangle)|^2 = \frac{1}{2}.$$

(d) The matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not unitary. This can be checked explicitly. For example

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I.$$

Thus, it can not model the absorption process of the photon by the atom.
The matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is unitary. Indeed

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus, it may model the absorption of the photon. Note also that this matrix acts like a Hadamard matrix on the subspace $\{|h\rangle\,, |abs\rangle\}$.

Exercice 2 *Un calcul qui sera utile pour l'algorithmique.*

(a) Prendre b=0 :

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\sum_{c=0}^{1}|c\rangle.$$

Prendre b=1 :

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\sum_{c=0}^{1}(-1)^c|c\rangle.$$

Pour $m = 2$:

$$\begin{aligned} H^{\otimes 2}|0,0\rangle = H \otimes H|00\rangle &= H \otimes H|0\rangle \otimes |0\rangle \\ &= H|0\rangle \otimes H|0\rangle \\ &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle}{2} \\ &= \frac{1}{(\sqrt{2})^2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle). \end{aligned}$$

En général pour $(b_1 b_2) = (00), (01), (10), (11)$:

$$\begin{aligned} H^{\otimes 2}|b_1 b_2\rangle &= H|b_1\rangle \otimes H|b_2\rangle \\ &= \frac{1}{\sqrt{2}}\sum_{c_1}(-1)^{b_1 c_1}|c_1\rangle \otimes \sum_{c_2}(-1)^{b_2 c_2}|c_2\rangle \\ &= (\frac{1}{\sqrt{2}})^2 \sum_{c_1,c_2}(-1)^{b_1 c_1}(-1)^{b_2 c_2}|c_1\rangle \otimes |c_2\rangle \\ &= (\frac{1}{\sqrt{2}})^2 \sum_{c_1,c_2}(-1)^{b_1 c_1 + b_2 c_2}|c_1 c_2\rangle. \end{aligned}$$

Pour $m = 3$ on procède de même et on généralise à m quelconque.

Exercice 3 *Représentations sur la sphere de Bloch.*

(a) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, thus from the formula given for the exponential of Pauli matrices

$$e^{i\frac{\theta}{2}\sigma_z} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$$

So $e^{i\frac{\theta}{2}\sigma_z}|\uparrow\rangle = e^{i\frac{\theta}{2}}|\uparrow\rangle$ and both vectors $|\uparrow\rangle$ and $e^{i\frac{\theta}{2}\sigma_z}|\uparrow\rangle$ are along z-axis. See fig.1

(b) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$e^{i\frac{\theta}{2}\sigma_x} = I\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\sigma_x = \begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

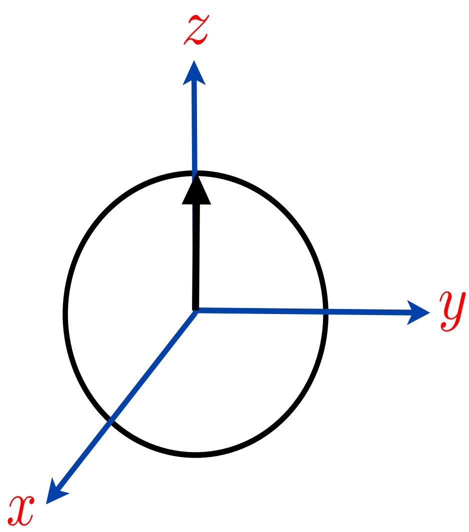

FIGURE 1 – fig.1

Therefore, we have

$$e^{i\frac{\theta}{2}\sigma_x}|\uparrow\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) \end{pmatrix} = \cos(\frac{\theta}{2})|\uparrow\rangle + i\sin(\frac{\theta}{2}|\downarrow\rangle$$
$$= \cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})|\downarrow\rangle$$

See fig 2.

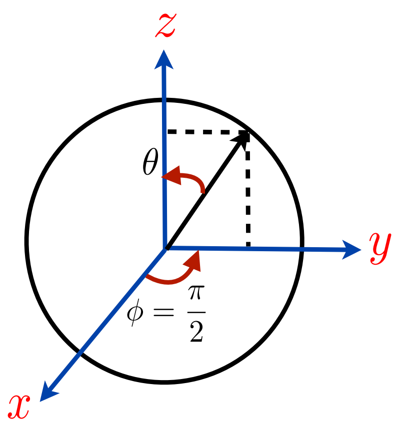

FIGURE 2 – fig.2

(c) Here $e^{i\frac{\theta}{2}\sigma_z} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$ and $\frac{|\uparrow\rangle+|\downarrow\rangle}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Then

$$e^{i\frac{\theta}{2}\sigma_z}(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}) = \frac{e^{i\frac{\theta}{2}}|\uparrow\rangle + e^{-i\frac{\theta}{2}}|\downarrow\rangle}{\sqrt{2}} = e^{i\frac{\theta}{2}}(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{e^{-i\theta}}{\sqrt{2}}|\downarrow\rangle).$$

We have a vector with $(\theta, \phi) = (\frac{\pi}{4}, -\theta)$. see fig 3.

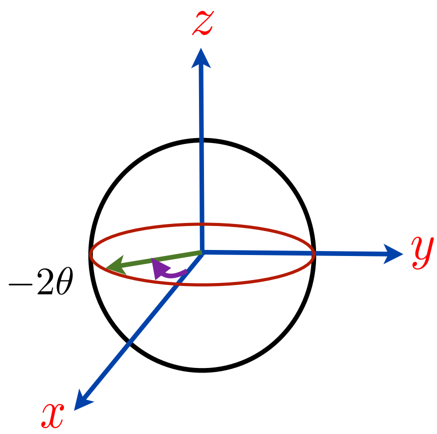

FIGURE 3 – fig.3

(d) We have

$$e^{it\frac{a}{2}\sigma_z}(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\phi}\sin(\frac{\theta}{2})\left|\downarrow\right\rangle) = \cos(\frac{\theta}{2})e^{it\frac{a}{2}}\left|\uparrow\right\rangle + e^{i\phi}\sin(\frac{\theta}{2})e^{-it\frac{a}{2}}\left|\downarrow\right\rangle$$
$$= e^{it\frac{a}{2}}(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i(\phi-ta)}\sin(\frac{\theta}{2})\left|\downarrow\right\rangle).$$

The angle θ stays fixed and the angle ϕ varies as $\phi \to \phi - 2ta$. The resulting trajectory is a circle around the z-axis where the period is given by $-2Ta = -2\pi$ which gives $T = \frac{\pi}{a}$. See fig 4.

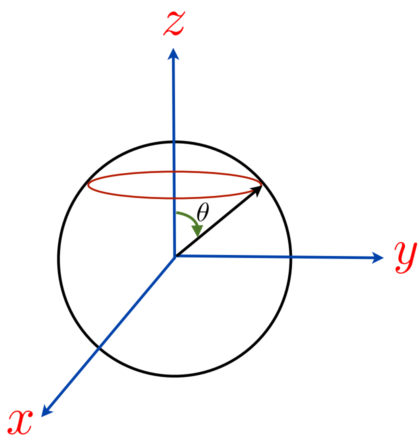

FIGURE 4 – fig 4.

Exercice 4 *Billets de banque quantiques.*

(a) La banque ne veut pas détruire le billet quantique. Donc elle doit faire des mesures dans des bases qui ne perturbe pas les états des photons. A partir du numéro de série S elle sait dans quelle base les états des photons ont été préparés. Elle fait donc une mesure de la polarisation dans la "bonne base“ : ainsi l'état qui est déja un des vecteurs de base ne change pas (ou est projeté sur lui même avec probabilité 1 !). La banque observe ainsi que tous les photons sont dans un état correct, et ceci sans détruire le billet.

(b) Le malfaiteur ne peut pas copier le billet avec une seule machine quantique (unitaire) à cause du théorème de non-clonage (no-cloning theorem). Notez qu'il est fondamental d'utiliser un ensemble non-orthogonal détats de photons pour préparer le billet de banque.

Exercice 5 *Un calcul utile pour les inégalités de Bell.*

1. $|\psi\rangle = |\gamma\rangle \otimes |\delta\rangle$. We use

$$\begin{aligned} A \otimes B &= (|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp|) \otimes (|\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp| \\ &= |\alpha\beta\rangle\langle\alpha\beta| - |\alpha\beta_\perp\rangle\langle\alpha\beta_\perp| - |\alpha_\perp\beta\rangle\langle\alpha_\perp\beta| + |\alpha_\perp\beta_\perp\rangle\langle\alpha_\perp\beta_\perp| \,. \end{aligned}$$

Pour obtenir l'égalité ci-dessus on utilise $|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| = (|\alpha\rangle \otimes |\beta\rangle)(\langle\alpha| \otimes \langle\beta|) = |\alpha\beta| \otimes \langle\alpha\beta|$. De plus on note que

$$\langle\gamma\delta|\alpha\beta\rangle\langle\alpha\beta|\gamma\delta\rangle = \langle\gamma|\alpha\rangle\langle\delta|\beta\rangle\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle = |\langle\alpha|\gamma\rangle|^2|\langle\beta|\delta\rangle|^2$$

On déduit

$$\begin{aligned} \langle\gamma,\delta|\, A \otimes B\, |\gamma,\delta\rangle &= |\,\langle\alpha|\,\gamma\rangle\,|^2|\,\langle\beta|\,\delta\rangle\,|^2 - |\,\langle\alpha|\,\gamma\rangle\,|^2|\,\langle\beta_\perp|\,\delta\rangle\,|^2 \\ &- |\,\langle\gamma|\,\alpha_\perp\rangle\,|^2|\,\langle\beta|\,\delta\rangle\,|^2 + |\,\langle\alpha_\perp|\,\gamma\rangle\,|^2|\,\langle\beta_\perp|\,\gamma\rangle\,|^2 \\ &= \cos^2(\alpha-\gamma)\cos^2(\beta-\delta) - \cos^2(\alpha-\gamma)\sin^2(\beta-\delta) \\ &- \sin^2(\gamma-\alpha)\cos^2(\beta-\delta) + \sin^2(\alpha-\gamma)\sin^2(\beta-\delta) \\ &= \cos^2(\alpha-\gamma)\cos(2(\beta-\delta)) - \sin^2(\alpha-\gamma)\cos(2(\beta-\delta)) \\ &= \cos(2(\alpha-\gamma))\cos(2(\beta-\delta)). \end{aligned}$$

2. Nous calculons la moyenne pour l'état de Bell $|\Psi\rangle = |B_{00}\rangle$. Comme vu dans une série précédente on peut exprimer létat de Bell comme $\frac{1}{\sqrt{2}}(|\alpha\alpha\rangle + |\alpha_\perp\alpha_\perp\rangle)$. Alors

$$\begin{aligned} \langle B_{00}|A \otimes B|B_{00}\rangle &= \frac{1}{2}\langle\alpha\alpha|A \otimes B|\alpha\alpha\rangle + \frac{1}{2}\langle\alpha_\perp\alpha_\perp|A \otimes B|\alpha_\perp\alpha_\perp\rangle \\ &+ \frac{1}{2}\langle\alpha\alpha|A \otimes B|\alpha_\perp\alpha_\perp\rangle + \frac{1}{2}\langle\alpha_\perp\alpha_\perp|A \otimes B|\alpha\alpha\rangle \\ &= \frac{1}{2}\langle\alpha|A|\alpha\rangle\langle\alpha|B|\alpha\rangle + \frac{1}{2}\langle\alpha_\perp|A|\alpha_\perp\rangle\langle\alpha_\perp|B|\alpha_\perp\rangle \\ &= \frac{1}{2}\cdot 1 \cdot \Big(|\langle\alpha|\beta\rangle|^2 - |\langle\alpha|\beta_\perp\rangle|^2\Big) + \frac{1}{2}\cdot(-1)\cdot\Big(|\langle\alpha_\perp|\beta\rangle|^2 - |\langle\alpha_\perp|\beta_\perp\rangle|^2\Big) \\ &= \frac{1}{2}\Big(\cos^2(\alpha-\beta) - \sin^2(\alpha-\beta)\Big) - \frac{1}{2}\Big(\sin^2(\alpha-\beta) - \cos^2(\alpha-\beta)\Big) \\ &= \cos^2(\alpha-\beta) - \sin^2(\alpha-\beta) = \cos 2(\alpha-\beta) \end{aligned}$$

Notez que les résultats sous 1) et 2) sont très différents. Cette différence est à la base des inégalités de Bell.

Exercice 6 *Heisenberg Uncertainty Principle.*

1. First we remark that $\Delta A' = \Delta A$ and $\Delta B' = \Delta B$. Moreover we check that $[A', B'] = [A, B]$. Thus it suffices to prove the inequality for observables that have zero mean.

2. Now assume the two observables have zero mean, and drop the primes for notational convenience. Let $|\phi_\lambda\rangle = (A + i\lambda B)|\psi\rangle$. For $\lambda \in \mathbb{R}$, let us define $f(\lambda) = \langle\phi_\lambda|\,\phi_\lambda\rangle$. It is clear that for any $\lambda \in \mathbb{R}$, $f(\lambda) \geq 0$. We also have

$$\begin{aligned} f(\lambda) &= \langle\psi|\,(A^\dagger - i\lambda^* B^\dagger)(A + i\lambda B)\,|\psi\rangle = \langle\psi|\,(A - i\lambda B)(A + i\lambda B)\,|\psi\rangle \\ &= \langle\psi|\,A^2\,|\psi\rangle + \lambda^2\,\langle\psi|\,B^2\,|\psi\rangle + i\lambda\,\langle\psi|\,(AB - BA)\,|\psi\rangle \\ &= \langle\psi|\,A^2\,|\psi\rangle + \lambda^2\,\langle\psi|\,B^2\,|\psi\rangle + \lambda\,\langle\psi|\,i[A,B]\,|\psi\rangle\,, \end{aligned}$$

where we used the Hermitian property of A and B and the fact that $\lambda \in \mathbb{R}$, thus $\lambda = \lambda^*$. First one can simply check that the operator $i[A,B]$ is a Hermitian operator so the last term $\langle\psi|\,i[A,B]\,|\psi\rangle$ is real-valued so $f(\lambda)$ is real-valued (this was already clear). We see that $f(\lambda)$ is a second order polynomial in $\lambda \in \mathbb{R}$. As it is non-negative for every value of λ, it results that its discriminant must be negative or zero (for a quadratic equation $ax^2 + bx + c = 0$ the discriminant is defined by $\Delta = b^2 - 4ac$). Hence, we get

$$|\,\langle\psi|\,[A,B]\,|\psi\rangle\,|^2 \leq 4\,\langle\psi|\,A^2\,|\psi\rangle\,\langle\psi|\,B^2\,|\psi\rangle\,.$$

As we assumed that the operators have zero mean, $\langle\psi|\,A\,|\psi\rangle = \langle\psi|\,B\,|\psi\rangle = 0$, we obtain that

$$\Delta A\,\Delta B \geq \sqrt{\frac{|\,\langle\psi|\,[A,B]\,|\psi\rangle\,|^2}{4}} = \frac{|\,\langle\psi|\,[A,B]\,|\psi\rangle\,|}{2}$$

3. For $\psi = |\uparrow\rangle$ and $A = \sigma_x$ and $B = \sigma_y$, we have

$$\begin{aligned} \bar{A} &= \langle\psi|\,A\,|\psi\rangle = (1\;\;0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0, \\ \bar{A^2} &= \langle\psi|\,A^2\,|\psi\rangle = (1\;\;0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1, \\ \bar{B} &= \langle\psi|\,B\,|\psi\rangle = (1\;\;0)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0, \\ \bar{B^2} &= \langle\psi|\,B^2\,|\psi\rangle = (1\;\;0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1, \end{aligned}$$

where for an operator C, $\bar{C}$ denotes the average of the operator in state $|\psi\rangle$. Therefore, we get

$$(\Delta A)^2 = \langle\psi|\,A^2\,|\psi\rangle - (\langle\psi|\,A\,|\psi\rangle)^2 = 1, (\Delta B)^2 = \langle\psi|\,B^2\,|\psi\rangle - (\langle\psi|\,B\,|\psi\rangle)^2 = 1.$$

We also know that the Pauli matrices satisfy the identity $[\sigma_x, \sigma_y] = 2i\sigma_z$. Thus, we have

$$\langle\psi|\,[A,B]\,|\psi\rangle = 2i(1\;\;0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 2i,$$

which satisfies the uncertainty principle

$$(\Delta A)^2(\Delta B)^2 = 1 \geq \frac{|2i|^2}{4} = \frac{|\,\langle\psi|\,[A,B]\,|\psi\rangle\,|^2}{4}.$$

In particular, in this case the uncertainty inequality turns out to be an equality.