Solution de la série 7
Traitement Quantique de l'Information

Exercice 1

1. One has to show that $\langle B_{x,y}|\, B_{x',y'}\rangle = \delta_{x,x'}\delta_{y,y'}$. We show it explicitly for two cases :

$$\begin{aligned}\langle B_{00}|\, B_{00}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle)\\ &= \frac{1}{2}(\langle 00|\, 00\rangle + \langle 00|\, 11\rangle + \langle 11|\, 00\rangle + \langle 11|\, 11\rangle).\end{aligned}$$

Now we have

$$\begin{aligned}\langle 00|\, 00\rangle = \langle 0|\, 0\rangle\, \langle 0|\, 0\rangle = 1, \langle 00|\, 11\rangle = \langle 0|\, 1\rangle\, \langle 0|\, 1\rangle = 0,\\ \langle 11|\, 00\rangle = \langle 1|\, 0\rangle\, \langle 1|\, 0\rangle = 0, \langle 11|\, 11\rangle = \langle 1|\, 1\rangle\, \langle 1|\, 1\rangle = 1.\end{aligned}$$

Thus we get that $\langle B_{00}|\, B_{00}\rangle = \frac{1}{2}(1+0+0+1) = 1$. Now let us consider

$$\begin{aligned}\langle B_{00}|\, B_{01}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle)\\ &= \frac{1}{2}(\langle 00|\, 01\rangle + \langle 00|\, 10\rangle + \langle 11|\, 01\rangle + \langle 11|\, 10\rangle)\\ &= \frac{1}{2}(0+0+0+0) = 0.\end{aligned}$$

2. The proof is by contradiction. Suppose there exist a_1, b_1 and a_2, b_2 such that

$$|B_{00}\rangle = (a_1\,|0\rangle + b_1\,|1\rangle) \otimes (a_2\,|0\rangle + b_2\,|1\rangle).$$

Then we must have

$$\frac{1}{2}(|00\rangle + |11\rangle) = a_1a_2\,|00\rangle + a_1b_2\,|01\rangle + b_1a_2\,|10\rangle + a_2b_2\,|11\rangle\,.$$

Since the states $|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle$ form a basis one has

$$\frac{1}{2} = a_1a_2,\ \frac{1}{2} = b_1b_2,\ a_1b_2 = 0,\ b_1a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $|B_{00}\rangle$ can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. Therefore, B_{00} is entangled.

3. We have

$$
\begin{aligned}
|\gamma\rangle \otimes |\gamma\rangle &= (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle) \otimes (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle) \\
&= \cos^2(\gamma)|00\rangle + \cos(\gamma)\sin(\gamma)|01\rangle + \sin(\gamma)\cos(\gamma)|10\rangle + \sin^2(\gamma)|11\rangle .
\end{aligned}
$$

Similarly,

$$
|\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = \cos^2(\gamma_\perp)|00\rangle + \cos(\gamma_\perp)\sin(\gamma_\perp)|01\rangle + \sin(\gamma_\perp)\cos(\gamma_\perp)|10\rangle + \sin^2(\gamma_\perp)|11\rangle .
$$

A picture shows that $\cos(\gamma_\perp) = -\sin(\gamma)$ and $\sin(\gamma_\perp) = \cos(\gamma)$ (this also allows to check that $\langle\gamma|\,\gamma_\perp\rangle = 0$). Therefore, $\cos^2(\gamma_\perp) = \sin^2(\gamma)$, $\sin^2(\gamma_\perp) = \cos^2(\gamma)$ and $\cos(\gamma_\perp)\sin(\gamma_\perp) = -\cos(\gamma)\sin(\gamma)$. We find that

$$
|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = (\cos^2(\gamma) + \sin^2(\gamma))|00\rangle + (\sin^2(\gamma) + \cos^2(\gamma))|11\rangle ,
$$

and the terms $|01\rangle$ and $|10\rangle$ cancel. Finally,

$$
\frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |B_{00}\rangle .
$$

4. From the rule for the tensor product

$$
\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} ,
$$

we get for the basis states

$$
|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ,
$$

$$
|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .
$$

Thus,

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},$$

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

Exercice 2

1. By definition of the tensor product :

$$(H \otimes I)\,|x\rangle \otimes |y\rangle = H\,|x\rangle \otimes I\,|y\rangle = H\,|x\rangle \otimes |y\rangle\,.$$

Also, one can use that $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ to show that always

$$H\,|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle).$$

Thus,

$$(H \otimes I)\,|x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x\,|1\rangle \otimes |y\rangle).$$

Now we apply *'CNOT'*. By linearity, we can apply it to each term separately. Thus,

$$\begin{aligned}(CNOT)(H \otimes I)\,|x\rangle \otimes |y\rangle &= \frac{1}{\sqrt{2}}((CNOT)\,|0\rangle \otimes |y\rangle + (-1)^x(CNOT)\,|1\rangle \otimes |y\rangle)\\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x\,|1\rangle \otimes |y \oplus 1\rangle)\\ &= |B_{xy}\rangle\,.\end{aligned}$$

2. Let us first start with $H \otimes I$. We use the rule

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} \otimes \begin{pmatrix}e & f\\g & h\end{pmatrix} = \begin{pmatrix}ae & af & be & bf\\ag & ah & bg & bh\\ce & cf & de & df\\cg & ch & dg & dh\end{pmatrix},$$

Thus we have

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}.$$

For (CNOT), we use the definition :

$$(CNOT)\,|x\rangle\otimes|y\rangle=|x\rangle\otimes|y\oplus x\rangle\,,$$

which implies that the matrix elements are

$$\langle x'y'|\,CNOT\,|xy\rangle=\langle x',y'|x,y\otimes x\rangle=\langle x'|\,x\rangle\,\langle y'|\,y\oplus x\rangle=\delta_{xx'}\delta_{y\oplus x,y'}.$$

We obtain the following table with columns xy and rows $x'y'$:

	00	01	10	11
00	1	0	0	0
01	0	1	0	0
10	0	0	0	1
11	0	0	1	0

For the matrix product $(CNOT)(H\otimes I)$, we find that

$$\begin{aligned}(CNOT)H\otimes I&=\frac{1}{\sqrt{2}}\begin{pmatrix}I & 0\\ 0 & X\end{pmatrix}\begin{pmatrix}I & I\\ I & -I\end{pmatrix}\\ &=\frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ X & -X\end{pmatrix},\end{aligned}$$

where $X=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Thus,

$$(CNOT)(H\otimes I)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & -1\\ 1 & 0 & -1 & 0\end{pmatrix}.$$

One can check that for example $|B_{00}\rangle=(CNOT)(H\otimes I)\,|0\rangle\otimes|0\rangle$. Finally to check the unitarity, we have to check that $UU^\dagger=U^\dagger U=I$ for $U=H\otimes I$, $CNOT$ and $(CNOT)(H\otimes I)$. We leave this to the reader.

3. Let $U=(CNOT)(H\otimes I)$. We have

$$|B_{xy}\rangle=U\,|x\rangle\otimes|y\rangle\,,\langle B_{x'y'}|=\langle x'|\otimes\langle y'|\,U^\dagger.$$

Thus,

$$\begin{aligned}\langle B_{x'y'}|\,B_{xy}\rangle&=\langle x'|\otimes\langle y'|\,U^\dagger U\,|x\rangle\otimes|y\rangle\\ &=\langle x'|\otimes\langle y'|\,I\,|x\rangle\otimes|y\rangle\\ &=\langle x'|\,x\rangle\,\langle y'|\,y\rangle=\delta_{xx'}\delta_{yy'}.\end{aligned}$$

Exercice 3

Read the course notes for a discussion of the solution.

Exercice 4

1. Read the calculations in the lecture notes.
2. $|\psi\rangle = |\gamma\rangle \otimes |\delta\rangle$. We use

$$\begin{aligned} A\otimes B &= (|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp|)\otimes(|\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp| \\ &= |\alpha\beta\rangle\langle\alpha\beta| - |\alpha\beta_\perp\rangle\langle\alpha\beta_\perp| - |\alpha_\perp\beta\rangle\langle\alpha_\perp\beta| + |\alpha_\perp\beta_\perp\rangle\langle\alpha_\perp\beta_\perp|\,. \end{aligned}$$

To get the above equality one uses $|\alpha\rangle\langle\alpha|\otimes|\beta\rangle\langle\beta| = (|\alpha\rangle\otimes|\beta\rangle)(\langle\alpha|\otimes\langle\beta|) = |\alpha\beta|\otimes\langle\alpha\beta|$. Furthermore note that

$$\langle\gamma\delta|\alpha\beta\rangle\langle\alpha\beta|\gamma\delta\rangle = \langle\gamma|\alpha\rangle\langle\delta|\beta\rangle\langle\alpha|\gamma\rangle\langle\beta|\delta\rangle = |\langle\alpha|\gamma\rangle|^2|\langle\beta|\delta\rangle|^2$$

One deduces

$$\begin{aligned} \langle\gamma,\delta|\,A\otimes B\,|\gamma,\delta\rangle &= |\,\langle\alpha|\,\gamma\rangle\,|^2|\,\langle\beta|\,\delta\rangle\,|^2 - |\,\langle\alpha|\,\gamma\rangle\,|^2|\,\langle\beta_\perp|\,\delta\rangle\,|^2 \\ &\quad - |\,\langle\gamma|\,\alpha_\perp\rangle\,|^2|\,\langle\beta|\,\delta\rangle\,|^2 + |\,\langle\alpha_\perp|\,\gamma\rangle\,|^2|\,\langle\beta_\perp|\,\gamma\rangle\,|^2 \\ &= \cos^2(\alpha-\gamma)\cos^2(\beta-\delta) - \cos^2(\alpha-\gamma)\sin^2(\beta-\delta) \\ &\quad - \sin^2(\gamma-\alpha)\cos^2(\beta-\delta) + \sin^2(\alpha-\gamma)\sin^2(\beta-\delta) \\ &= \cos^2(\alpha-\gamma)\cos(2(\beta-\delta)) - \sin^2(\alpha-\gamma)\cos(2(\beta-\delta)) \\ &= \cos(2(\alpha-\gamma))\cos(2(\beta-\delta)). \end{aligned}$$

3. The correlation X introduced in the notes is

$$\begin{aligned} X &= \cos(2(\alpha-\gamma))\cos(2(\beta-\delta) + \cos(2(\alpha-\gamma))\cos(2(\beta'-\delta)) \\ &\quad - \cos(2(\alpha'-\gamma))\cos(2(\beta-\delta)) + \cos(2(\alpha'-\gamma))\cos(2(\beta'-\delta)). \end{aligned}$$

one can rewrite it as

$$\begin{aligned} X &= \cos(2(\alpha-\gamma))\{\cos(2(\beta-\delta)) + \cos(2(\beta'-\delta))\} \\ &\quad - \cos(2(\alpha'-\gamma))\{\cos(2(\beta-\delta)) - \cos(2(\beta'-\delta))\}. \end{aligned}$$

this is of the form $X = p(\ell+r) - q(\ell-r)$ with p,q,ℓ,r in $[-1,1]$. To show that $|X|\leq 2$, one can proceed as follows. First fix p and q (i.e. fix α and α') and maximize or minimize over ℓ and r (i.e. over β and β'). Since X is linear in terms of ℓ and r, the maximum or minimum is achieved at boundary points of $\ell,r\in[-1,1]$. Thus we can assume that $\ell,r\in\{-1,+1\}$. Then we are back to the argument that we we have in the course notes : if $\ell=r=\pm1$, we have $X=\pm2p$ thus $|X|\leq 2$. If $\ell\neq r$, we have $X=\pm2q$ and again $|X|\leq 2$.

Remark : We see that for the product state $|X|\leq 2$. Finding, $|X|>2$ is a signature of entanglement. The value of $|X|$ for a given state $|\psi\rangle$ is a certicate of entanglement : if $|X|\leq 2$ the state is a product state; if $|X|>2$ the state is an entangled state. One can show that it is always the case that $|X|\leq 2\sqrt{2}$. In this sense the bell state is the “most entangled” state of a pair of qubits. Finding such certificates for arbitrary numbers of qubits is an interesting problem.