Solution de la série 5
Traitement Quantique de l'Information

Exercice 1 *Energy levels of a magnetic moment in a constant field.*

1. The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, the Hamiltonian becomes

$$\begin{aligned} H &= -\gamma(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z) \\ &= -\gamma \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}. \end{aligned}$$

To find the eigen-values, we should solve the equation $|H - \lambda I| = 0$, where I denotes the identity matrix of order 2 and $|.|$ is the determinant operator. For simplicity, we assume that $\gamma = 1$. Simplifying the equation $|H - \lambda I|$, we obtain that

$$-(B_z - \lambda)(B_z + \lambda) - (B_x + iB_y)(B_x - iB_y) = 0,$$

which implies that

$$\lambda^2 - B_z^2 - (B_x^2 + B_y^2) = 0.$$

Therefore, we get $\lambda = \pm B$ where $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$ is the Euclidean norm of the vector (B_x, B_y, B_z). The energy levels of the magnetic moment are $E_\pm = \pm\gamma B$.

To find the eigen-states (eigen-vectors), we solve $H\,|\psi_\pm\rangle = E_\pm\,|\psi_\pm\rangle$. Set $|\psi_\pm\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. We have

$$\begin{cases} -\gamma B_z \alpha - \gamma(B_x - iB_y)\beta = E_\pm \alpha, \\ -\gamma(B_x + iB_y)\alpha + \gamma B_z \beta = E_\pm \beta. \end{cases}$$

This implies that

$$(E_\pm + \gamma B_z)\alpha = -\gamma(B_x - iB_y)\beta \;\Rightarrow\; \beta = -\frac{E_\pm + \gamma B_z}{\gamma(B_x - iB_y)}\alpha.$$

Therefore, two eigen-vectors can be obtained by

$$|\psi_\pm\rangle = \alpha_\pm \begin{pmatrix} 1 \\ -\frac{\pm B + B_z}{B_x^2 + B_y^2}(B_x + iB_y) \end{pmatrix} = \alpha_\pm \left(|\uparrow\rangle - \frac{\pm B + B_z}{B_x^2 + B_y^2}(B_x + iB_y)\,|\downarrow\rangle \right),$$

where $\alpha_\pm$ can be obtained from the normalization condition to be $\alpha_\pm = 1/\sqrt{1+\frac{(\pm B+B_z)^2}{B_x^2+B_y^2}}$. To represent the eigen-vectors on the Bloch sphere, we note that if $|\psi\rangle = \cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\phi}\sin(\frac{\theta}{2})|\downarrow\rangle$, we can simply identify that

$$e^{i\phi} = \frac{B_x + iB_y}{\sqrt{B_x^2+B_y^2}}, \ \tan(\frac{\theta}{2}) = -\frac{\pm B + B_z}{\sqrt{B_x^2+B_y^2}}.$$

On the Bloch sphere, the direction of this state is along θ, ϕ as shown in Figure 1.

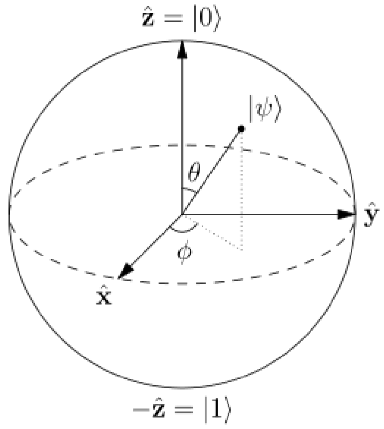

FIGURE 1 – Bloch Sphere.

Let us show that in fact this is the direction of the vector $\vec{B}$. Let assume that the direction of $\vec{B}$ is denoted by two angles ζ, η. Then, we have

$$\cos(\eta) = \frac{B_x}{\sqrt{B_x^2+B_y^2}}, \ \sin(\eta) = \frac{B_y}{\sqrt{B_x^2+B_y^2}} \ \Rightarrow e^{i\eta} = \frac{B_x+iB_y}{\sqrt{B_x^2+B_y^2}} = e^{i\phi},$$

which implies that $\eta = \phi$. Also, $\tan(\zeta) = \frac{\sqrt{B_x^2+B_y^2}}{B_z}$. Now we need to show that $\tan(\theta) = \tan(\zeta)$. Using the identity $\tan(\theta) = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$, one gets that

$$\begin{aligned}
\tan(\theta) &= -\frac{2(\pm B+B_z)}{\sqrt{B_x^2+B_y^2}} \frac{1}{1-\frac{(\pm B+B_z)^2}{B_x^2+B_y^2}} \\
&= -\frac{2(\pm B+B_z)}{\sqrt{B_x^2+B_y^2}} \frac{B_x^2+B_y^2}{B_x^2+B_y^2-(\pm B+B_z)^2} \\
&= -\frac{2(\pm B+B_z)\sqrt{B_x^2+B_y^2}}{-2B_z(\pm B+B_z)} \\
&= \frac{\sqrt{B_x^2+B_y^2}}{B_z} = \tan(\zeta).
\end{aligned}$$

Conclusion : On the Bloch sphere, the eigen-states are in the direction of $\vec{B}$; one along $\vec{B}$ and the other opposite to $\vec{B}$.

2. To obtain the results without any calculations, assume that we have taken the z axis along the vector $\vec{B}$ (this does not change the physical results because H is rotation invariant !). In the new coordinate, we have the following Hamiltonian

$$\tilde{H} = -\gamma B \sigma_z = \begin{pmatrix} -\gamma B & 0 \\ 0 & \gamma B \end{pmatrix}.$$

The eigen-values are simply $E_\pm = \pm\gamma B$ (as before !) and the new eigen-states are

$$|\psi_-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |\psi_+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where one of them is along $\vec{B}$ and the other one opposite to $\vec{B}$ (as before !).

3. For the transition $|\downarrow\rangle \to |\uparrow\rangle$, $\Delta E = E_{\text{photon}} = E_+ - E_- = 2\gamma B$, thus $f_{\text{photon}} = \frac{2\gamma B}{h}$(HZ) where h denotes Planck's constant.

Exercice 2 *Stern-Gerlach apparatus*

1. Given that the magnetic moment is in the state $|\theta, \phi\rangle = \cos(\theta/2)\,|\uparrow\rangle + e^{i\phi}\sin(\theta/2)\,|\downarrow\rangle$, the probability of obtaining $|\uparrow\rangle$ is given by

$$|\,\langle\uparrow|\,\theta,\phi\rangle\,|^2 = |\,\langle\uparrow\big|\cos(\theta/2)\,|\uparrow\rangle + e^{i\phi}\sin(\theta/2)\,|\downarrow\rangle\,\rangle\,|^2 = \cos^2(\theta/2).$$

Similarly, the probability of obtaining $|\downarrow\rangle$ is $\sin^2(\theta/2)$.

2. If $p(\theta,\phi) = \frac{1}{4\pi}$, the total number of particles observed with $|\uparrow\rangle$ is given by

$$N_\uparrow = \frac{N}{4\pi}\int_{-1}^{1} d(\cos(\theta)) \int_0^{2\pi} d\phi \cos^2(\theta/2) = \frac{N}{2}\int_{-1}^{1} d(\cos(\theta))\cos^2(\theta/2).$$

Using the identity $\cos(\theta) = 2\cos^2(\theta/2) - 1$, one gets

$$N_\uparrow = \frac{N}{2}\int_{-1}^{1} d(\cos(\theta))\frac{1+\cos(\theta)}{2} = \frac{N}{8}(1+x)^2|_{-1}^{1} = \frac{N}{2}.$$

A similar calculation by replacing $\cos^2(\theta/2)$ with $\sin^2(\theta/2)$ gives $N_\downarrow = \frac{N}{2}$. Thus $N_\uparrow + N_\downarrow = N$ as expected.

3. If $p(\theta,\phi) = \frac{1}{2\pi}$ for $0 \le \theta \le \frac{\pi}{2}$ and zero elsewhere, we have

$$N_\uparrow = \frac{N}{2\pi}\int_0^1 d(\cos(\theta)) \int_0^{2\pi} d\phi \cos^2(\theta/2) = \frac{N}{2}\int_0^1 d(\cos(\theta))\cos^2(\theta/2) = \frac{3N}{4}.$$

A similar computation gives $N_\downarrow = \frac{N}{4}$. We see that if initially all magnetic moments are polarized in the upper hemisphere, there is a portion of them that is observed in the lower part of the screen (of the SG experiment).

Exercice 3 *"Dynamique" probabiliste versus quantique*

1. Pour montrer que $|\langle j| U |i\rangle|^2 = R_{ji}$ est une matrice stochastique, il faut verifier que $\sum_{j=0}^{n-1} P_{ji} = 1$ et $0 \leq P_{ji} \leq 1$:

$$\begin{aligned}\sum_{j=0}^{n-1} P_{ji} &= \sum_{j=0}^{n-1} |\langle j| U |i\rangle|^2 = \sum_{j=0}^{n-1} \langle i| U^\dagger |j\rangle \langle j| U |i\rangle \\ &= \langle i| U^\dagger \sum_{j=0}^{n-1} |j\rangle \langle j| U |i\rangle = \langle i| U^\dagger U |i\rangle \\ &= \langle i| i\rangle = 1\end{aligned}$$

   Notez que tous les termes $|\langle j| U |i\rangle|^2$ sont positifs, c'est pourquoi $0 \leq P_{ji} \leq 1$.

2. La probabilité d'observer l'état 2 à la sortie est

$$\text{Prob}(2) = P_{00}Q_{20} + P_{10}Q_{21} + P_{20}Q_{22}.$$

3. – La probabilité d'observer l'état $|2\rangle$ à la sortie est

$$\begin{aligned}\text{Prob}(|2\rangle) &= |\langle 2| U_2U_1 |0\rangle|^2 = \langle 2| U_2U_1 |0\rangle \langle 0| U_1^\dagger U_2^\dagger |2\rangle \\ &= \sum_{i=0}^{2}\sum_{j=0}^{2} \langle 2| U_2 |i\rangle \langle i| U_1 |0\rangle \langle 0| U_1^\dagger |j\rangle \langle j| U_2^\dagger |2\rangle \\ &= \sum_{i=j} \langle 2| U_2 |i\rangle \langle i| U_1 |0\rangle \langle 0| U_1^\dagger |i\rangle \langle i| U_2^\dagger |2\rangle \\ &+ \sum_{i\neq j} \langle 2| U_2 |i\rangle \langle i| U_1 |0\rangle \langle 0| U_1^\dagger |j\rangle \langle j| U_2^\dagger |2\rangle\end{aligned}$$

   Notez que la première somme est la probabilité d'observer l'état 2 à la sortie dans le cas classique :

$$\begin{aligned}\sum_{i=0}^{2} \langle 2| U_2 |i\rangle \langle i| U_1 |0\rangle \langle 0| U_1^\dagger |i\rangle \langle i| U_2^\dagger |2\rangle &= \sum_{i=0}^{2} |\langle 2| U_2 |i\rangle|^2 |\langle i| U_1 |0\rangle|^2 \\ &= P_{00}Q_{20} + P_{10}Q_{21} + P_{20}Q_{22}\end{aligned}$$

   La deuxième somme est un terme d'interférence quantique.

   – En faisant la mesure intermédiaire après la première étape on observe
   - $|0\rangle$ avec la probabilité $|\langle 0| U_1 |0\rangle|^2$,
   - $|1\rangle$ avec la probabilité $|\langle 1| U_1 |0\rangle|^2$,
   - $|2\rangle$ avec la probabilité $|\langle 2| U_1 |0\rangle|^2$.
   La probabilité d'observer l'état final $|2\rangle$ à la sortie est

$$|\langle 0| U_1 |0\rangle|^2 |\langle 2| U_2 |0\rangle|^2 + |\langle 1| U_1 |0\rangle|^2 |\langle 2| U_2 |1\rangle|^2 + |\langle 2| U_1 |0\rangle|^2 |\langle 2| U_2 |2\rangle|^2$$

   qui correspond au cas classique.