Solution de la série 13
Traitement quantique de l'information

Exercice 1 *Refocusing*

– One way is to find the corresponding matrices and multiply them together to show the identity. A probably simpler way is to show that for every basis vector both right hand side and left hand side operators give the same result. For example for $|\psi_0\rangle = |\uparrow\uparrow\rangle$, applying the matrices starting form the right hand side one, one obtains (using that R_1 flips a spin ; check this !)

$$|\psi_1\rangle = e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}} |\uparrow\uparrow\rangle = e^{-itJ} |\psi_0\rangle ,$$
$$|\psi_2\rangle = (R_1 \otimes I) |\psi_1\rangle = e^{-itJ} |\downarrow\uparrow\rangle ,$$
$$|\psi_3\rangle = e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}} |\psi_2\rangle = e^{-itJ} e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}} |\downarrow\uparrow\rangle = e^{-itJ} e^{itJ} |\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle ,$$
$$|\psi_4\rangle = (R_1 \otimes I) |\psi_3\rangle = (R_1 \otimes I) |\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle ,$$

which shows that $|\psi_4\rangle = |\psi_0\rangle = (I_1 \otimes I_2) |\psi_0\rangle$. One can also check this for other basis vectors to see that the identity indeed holds.

– $J << 1$. Donc $\tau = \frac{\pi}{4J} >> \pi$. Les π-pulses sont beaucoup plus rapides que l'évolution des spins nucleaires. L'idée est que en injectant deux π-pulses aux instants $\frac{\tau}{2}$ et τ on reforme l'état initial et donc tout se passe comme si les deux spins n'avaient pas évolué.

Exercice 2 *Realization of the SWAP gate*

– To find the matrix representation, it is sufficient to find how SWAP port operates on the basis vectors.

$$\text{SWAP} |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

$$\text{SWAP} |\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ,$$

$$\text{SWAP} |\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ,$$

$$\text{SWAP} |\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

Putting the resulting columns together we obtain the matrix representation

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now it is easy to check that $(\text{SWAP})(\text{SWAP}^\dagger) = I$ which shows that SWAP is a unitary matrix.

– The Heisenberg Hamiltonian is obtained in the lecture notes and has the following matrix representation

$$\mathcal{H} = \hbar J \vec{\sigma_1}.\vec{\sigma_2} = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To compute the evolution operator $e^{-\frac{it\mathcal{H}}{\hbar}}$, we notice that the matrix for $\mathcal{H}$ has the diagonal representation

$$\begin{pmatrix} A & & \mathbf{0} \\ & B & \\ \mathbf{0} & & C \end{pmatrix},$$

where $A = (1)$ and $C = (1)$ are 1×1 matrices and $B = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ is a 2×2 matrix. It is easy to show that for any complex number α

$$e^{\alpha\mathcal{H}} = \begin{pmatrix} e^{\alpha A} & & \mathbf{0} \\ & e^{\alpha B} & \\ \mathbf{0} & & e^{\alpha C} \end{pmatrix}.$$

Thus it is sufficient to find these three matrix exponentials. A and C are numbers equal to 1, thus $e^{\alpha A} = e^{\alpha C} = e^{\alpha}$.

Now it remains to find $e^{\alpha B}$. Notice that we can write $B = -I + 2X$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that I and X commute with each other, i.e., $IX = XI$. It is not difficult to show that the matrices that commute with each other can be treated like numbers while taking exponentials, namely, for any commuting matrix M, N, $e^{M+N} = e^M e^N$. **(Notice that this formula is not in general correct)**. Therefore, we have

$$e^{i\beta B} = e^{-i\beta I} e^{2i\beta X} = e^{-i\beta}(I\cos(2\beta) + iX\sin(2\beta)),$$

where we used the Euler's formula for X. It can be checked that at time $t = \frac{\pi}{4J}$, $\alpha = -i\frac{\pi}{4}$, thus $\beta = -\frac{\pi}{4}$. Hence, $e^{\alpha A} = e^{\alpha B} = e^{-i\frac{\pi}{4}}$, and

$$e^{i\beta B} = e^{i\frac{\pi}{4}}(\cos(\frac{\pi}{2})I - i\sin(\frac{\pi}{2})X) = -ie^{i\frac{\pi}{4}}X = e^{-i\frac{\pi}{4}}X.$$

Putting all together, the evolution operator at time $t = \frac{\pi}{4J}$ is

$$e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which neglecting the constant phase $-\frac{\pi}{4}$ is equal to the matrix for SWAP.

– We can implement SWAP using three CNOT gates as depicted in Figure 1. To show this, one can simply check that starting from a general state $|x, y\rangle$, with $x, y \in \{0, 1\}$ after the first CNOT the resulting state is $|x, y \oplus x\rangle$, after the second CNOT the state is

$$|x \oplus (x \oplus y), x \oplus y\rangle = |x \oplus x \oplus y, x \oplus y\rangle = |y, x \oplus y\rangle \, ,$$

where we used the identity $x \oplus x = 0$ for $x \in \{0, 1\}$. Finally after the third CNOT the state is $|y, (x \oplus y) \oplus y\rangle = |y, x\rangle$. Therefore the combination the three gates just swaps x and y. Note that this gives another proof that SWAP is a unitary matrix because it can be implemented as a combination of quantum gates and we know that all quantum gates are unitary.

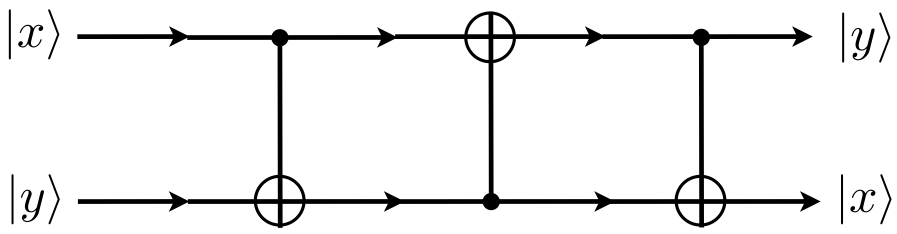

FIGURE 1 – Implementation of SWAP gate using three CNOT