Solution de la série 9
Traitement quantique de l'information II

Exercice 1 *Refocusing*

L'état final est

$$\begin{aligned}
e^{-\frac{it}{\hbar}\mathcal{H}}|\psi_0\rangle &= e^{-itJ\sigma_1^z\otimes\sigma_2^z}\cdot\frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)\\
&= \frac{1}{2}\left(e^{-itJ}|\uparrow\uparrow\rangle - e^{itJ}|\uparrow\downarrow\rangle + e^{itJ}|\downarrow\uparrow\rangle - e^{-itJ}|\downarrow\downarrow\rangle\right)\\
&= \frac{e^{-itJ}}{2}\left(|\uparrow\uparrow\rangle - e^{2itJ}|\uparrow\downarrow\rangle + e^{2itJ}|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right).
\end{aligned}$$

– Pour $t = \frac{\pi}{4J}$ on a $e^{2itJ} = e^{\frac{i\pi}{2}} = i$
$\Rightarrow |\psi_\tau\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$.

– Supposons que l'état puisse s'écrire
$(\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle)\otimes(\gamma\,|\uparrow\rangle + \delta\,|\downarrow\rangle) = \alpha\gamma\,|\uparrow\uparrow\rangle + \alpha\delta\,|\uparrow\downarrow\rangle + \beta\gamma\,|\downarrow\uparrow\rangle + \beta\delta\,|\downarrow\downarrow\rangle$,
alors $\alpha\gamma = 1,\ \alpha\delta = -i,\ \beta\gamma = i,\ \beta\delta = -1$.
On peut toujours poser $\alpha = 1$ (phase globale). Donc $\gamma = 1,\ \delta = -i,\ \beta = i$ et $\delta = i \Rightarrow$ contradiction sur δ. Vous pouvez aussi prendre n'importe quelle valeur fixée pour α et montrer qu'une contradiction apparait.

Exercice 2 *Refocusing*

– One way is to find the corresponding matrices and multiply them together to show the identity. A probably simpler way is to show that for every basis vector both right hand side and left hand side operators give the same result. For example for $|\psi_0\rangle = |\uparrow\uparrow\rangle$, applying the matrices starting form the right hand side one, one obtains (using that R_1 flips a spin ; check this !)

$$\begin{aligned}
&|\psi_1\rangle = e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}|\uparrow\uparrow\rangle = e^{-itJ}|\psi_0\rangle\,,\\
&|\psi_2\rangle = (R_1\otimes I)\,|\psi_1\rangle = e^{-itJ}|\downarrow\uparrow\rangle\,,\\
&|\psi_3\rangle = e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}|\psi_2\rangle = e^{-itJ}e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}|\downarrow\uparrow\rangle = e^{-itJ}e^{itJ}|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle\,,\\
&|\psi_4\rangle = (R_1\otimes I)\,|\psi_3\rangle = (R_1\otimes I)\,|\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle\,,
\end{aligned}$$

which shows that $|\psi_4\rangle = |\psi_0\rangle = (I_1\otimes I_2)\,|\psi_0\rangle$. One can also check this for other basis vectors to see that the identity indeed holds.

– $J << 1$. Donc $\tau = \frac{\pi}{4J} >> \pi$. Les π-pulses sont beaucoup plus rapides que l'évolution des spins nucleaires. L'idée est que en injectant deux π-pulses aux instants $\frac{\tau}{2}$ et τ on reforme l'état initial et donc tout se passe comme si les deux spins n'avaient pas évolué.

Exercice 3 *Realization of the SWAP port*

– To find the matrix representation, it is sufficient to find how SWAP port operates on the basis vectors.

$$\mathrm{SWAP}\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$

$$\mathrm{SWAP}\,|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},$$

$$\mathrm{SWAP}\,|\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

$$\mathrm{SWAP}\,|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Putting the resulting columns together we obtain the matrix representation

$$\mathrm{SWAP} = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Now it is easy to check that $(\mathrm{SWAP})(\mathrm{SWAP}^\dagger) = I$ which shows that SWAP is a unitary matrix.

– The Heisenberg Hamiltonian is obtained in the lecture notes and has the following matrix representation

$$\mathcal{H} = \hbar J \vec{\sigma_1}.\vec{\sigma_2} = \hbar J \begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}.$$

To compute the evolution operator $e^{-\frac{it\mathcal{H}}{\hbar}}$, we notice that the matrix for $\mathcal{H}$ has the diagonal representation

$$\begin{pmatrix}A & & \mathbf{0}\\ & B & \\ \mathbf{0} & & C\end{pmatrix},$$

where $A = (1)$ and $C = (1)$ are 1×1 matrices and $B = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ is a 2×2 matrix. It is easy to show that for any complex number α

$$e^{\alpha \mathcal{H}} = \begin{pmatrix} e^{\alpha A} & & \mathbf{0} \\ & e^{\alpha B} & \\ \mathbf{0} & & e^{\alpha C} \end{pmatrix}.$$

Thus it is sufficient to find these three matrix exponentials. A and C are numbers equal to 1, thus $e^{\alpha A} = e^{\alpha C} = e^{\alpha}$.

Now it remains to find $e^{\alpha B}$. Notice that we can write $B = -I + 2X$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that I and X commute with each other, i.e., $IX = XI$. It is not difficult to show that the matrices that commute with each other can be treated like numbers while taking exponentials, namely, for any commuting matrix M, N, $e^{M+N} = e^M e^N$. **(Notice that this formula is not in general correct)**. Therefore, we have

$$e^{i\beta B} = e^{-i\beta I} e^{2i\beta X} = e^{-i\beta}(I\cos(2\beta) + iX\sin(2\beta)),$$

where we used the Euler's formula for X. It can be checked that at time $t = \frac{\pi}{4J}$, $\alpha = -i\frac{\pi}{4}$, thus $\beta = -\frac{\pi}{4}$. Hence, $e^{\alpha A} = e^{\alpha B} = e^{-i\frac{\pi}{4}}$, and

$$e^{i\beta B} = e^{i\frac{\pi}{4}}(\cos(\frac{\pi}{2})I - i\sin(\frac{\pi}{2})X) = -ie^{i\frac{\pi}{4}}X = e^{-i\frac{\pi}{4}}X.$$

Putting all together, the evolution operator at time $t = \frac{\pi}{4J}$ is

$$e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which neglecting the constant phase $-\frac{\pi}{4}$ is equal to the matrix for SWAP.

– We can implement SWAP using three CNOT gates as depicted in Figure 1. To show this, one can simply check that starting from a general state $|x, y\rangle$, with $x, y \in \{0, 1\}$ after the first CNOT the resulting state is $|x, y \oplus x\rangle$, after the second CNOT the state is

$$|x \oplus (x \oplus y), x \oplus y\rangle = |x \oplus x \oplus y, x \oplus y\rangle = |y, x \oplus y\rangle,$$

where we used the identity $x \oplus x = 0$ for $x \in \{0, 1\}$. Finally after the third CNOT the state is $|y, (x \oplus y) \oplus y\rangle = |y, x\rangle$. Therefore the combination the three gates just swaps x and y.

Note that this gives another proof that SWAP is a unitary matrix because it can be implemented as a combination of quantum gates and we know that all quantum gates are unitary.

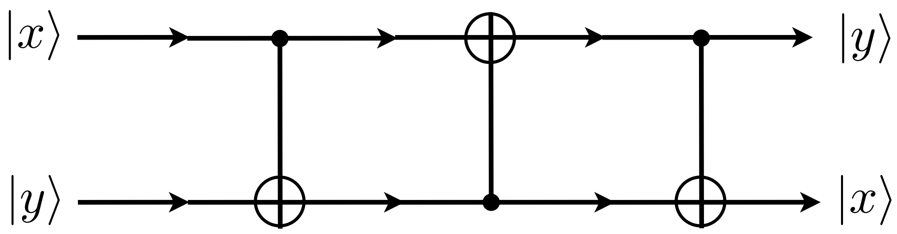

Figure 1 – Implementation of SWAP gate using three CNOT