Solution de la série 11
Traitement Quantique de l'Information

Exercice 1 *Review of the Bloch sphere*

A general vector can be written in the form $\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\phi}|\downarrow\rangle$ in the Bloch sphere.

- The bases vectors for the Z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$ which correspond to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$ respectively.
- The bases vectors for the Y are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle$ which correspond to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{pi}{2})$.
- Bases vectors for the X basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$ corresponding to $(\theta = \frac{\pi}{2}, \phi = \pi)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$ respectively.

The corresponding representation over the Bloch sphere is shown in Figure 1.

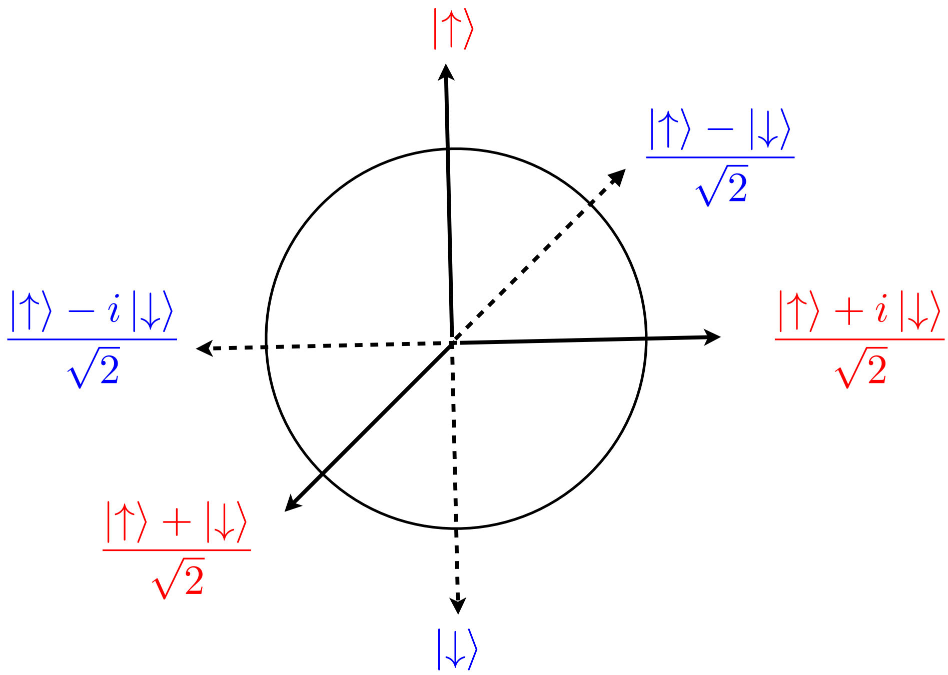


FIGURE 1 – Representation of basis vectors on Bloch Sphere

Using the general formula

$$\exp(i\frac{\theta}{2}(\vec{\sigma}.\vec{n})) = \cos(\frac{\theta}{2})I + i\vec{\sigma}.\vec{n}(\sin(\frac{\theta}{2})),$$

we obtain that

$$\begin{aligned}\exp(i\frac{\alpha}{2}\sigma_x) &= \cos(\frac{\alpha}{2})I + i\sigma_x(\sin(\frac{\alpha}{2}))\\ &= \begin{pmatrix}\cos(\frac{\alpha}{2}) & i\sin(\frac{\alpha}{2})\\ i\sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2})\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(i\frac{\beta}{2}\sigma_y) &= \cos(\frac{\beta}{2})I + i\sigma_y(\sin(\frac{\beta}{2}))\\ &= \begin{pmatrix}\cos(\frac{\beta}{2}) & \sin(\frac{\beta}{2})\\ -\sin(\frac{\beta}{2}) & \cos(\frac{\beta}{2})\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(i\frac{\gamma}{2}\sigma_z) &= \cos(\frac{\gamma}{2})I + i\sigma_z(\sin(\frac{\gamma}{2}))\\ &= \begin{pmatrix}\cos(\frac{\gamma}{2}) + i\sin(\frac{\gamma}{2}) & 0\\ 0 & \cos(\frac{\gamma}{2}) - i\sin(\frac{\gamma}{2})\end{pmatrix}\\ &= \begin{pmatrix}e^{i\frac{\gamma}{2}} & 0\\ 0 & e^{-i\frac{\gamma}{2}}\end{pmatrix}.\end{aligned}$$

The matrix $\exp(i\frac{\alpha}{2}\sigma_x)$ is a rotation matrix of angle α around the X-axis, thus the state vector $\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})\left|\downarrow\right\rangle$ will transform to the vector $\cos(\frac{\theta+\alpha}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta+\alpha}{2})\left|\downarrow\right\rangle$. One can see that geometrically over the Bloch sphere, however one can also show by direct calculation that

$$\exp(-i\frac{\alpha}{2}\sigma_x)\left(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})\right)\left|\downarrow\right\rangle = \cos(\frac{\theta+\alpha}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta+\alpha}{2})\left|\downarrow\right\rangle.$$

Similarly, one can see that $\exp(i\frac{\gamma}{2}\sigma_z)$ is a rotation of angle γ around the Z-axis. Therefore,

$$\exp(-i\frac{\gamma}{2}\sigma_z)\left(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})\right)\left|\downarrow\right\rangle = e^{-i\frac{\gamma}{2}}\left(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin(\frac{\theta}{2})\left|\downarrow\right\rangle\right).$$

Exercice 2 *Hamiltonian of the interaction of two spin $\frac{1}{2}$ particles*

– In the canonical bases, we have $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Using the tensor product rule one obtains that

$$\begin{aligned}\sigma_1^z \otimes \sigma_2^z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\end{aligned}$$

thus the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix}\hbar J & 0 & 0 & 0\\ 0 & -\hbar J & 0 & 0\\ 0 & 0 & -\hbar J & 0\\ 0 & 0 & 0 & \hbar J\end{pmatrix}.$$

– In the bra-ket formalism one has $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, thus

$$\begin{aligned}\sigma_1^z \otimes \sigma_2^z &= (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \\ &= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| .\end{aligned}$$

Therefore, we have

$$= \hbar J(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|).$$

Notice that to verify this one can use

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix},$$

which implies that

$$(|\uparrow\rangle\langle\uparrow|) \otimes (|\uparrow\rangle\langle\uparrow|) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Similarly one can show that

$$|\uparrow\downarrow\rangle\langle\uparrow\downarrow| = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$|\downarrow\uparrow\rangle\langle\downarrow\uparrow| = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},$$

$$|\downarrow\downarrow\rangle\langle\downarrow\downarrow| = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

– One can see that the eigen-values are $\hbar J$ corresponding to the eigen-vectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and $-\hbar J$ corresponding to the eigen-vectors $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$.

Exercice 3 *Identité utile pour la réalisation expérimentale de la porte CNOT par RMN*

– $\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ est diagonale donc

$$R_1 = R_2 = \begin{pmatrix}\exp(-i\frac{\pi}{4}) & 0\\0 & \exp(i\frac{\pi}{4})\end{pmatrix} = e^{-i\frac{\pi}{4}}\begin{pmatrix}1 & 0\\0 & i\end{pmatrix}.$$

– La porte de Hadamard est comme d'habitude $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$.

– Pour l'Hamiltonien on a :

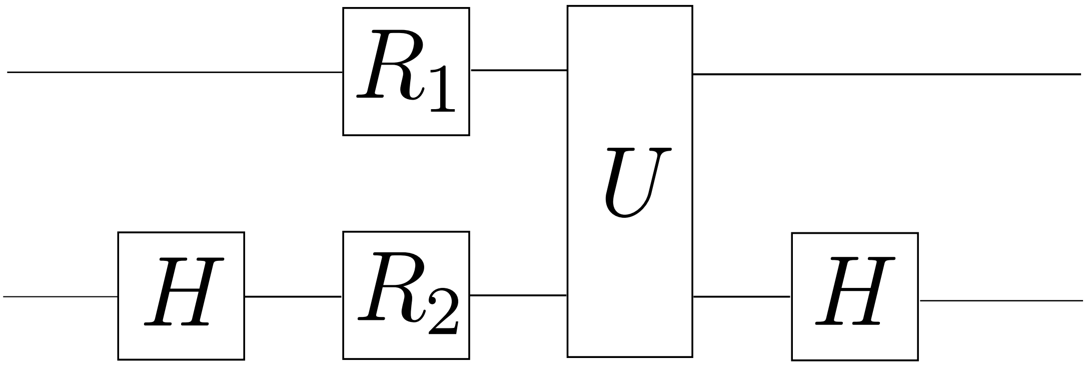
$$\mathcal{H} = \hbar J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

– Si on laisse évoluer pendant un temps $t = \frac{\pi}{4J}$ on trouve

$$U = \exp(-\frac{it}{\hbar}\mathcal{H}) = \exp(-\frac{i\pi}{4J\hbar}\mathcal{H}) = \begin{pmatrix} \exp(-i\frac{\pi}{4}) & 0 & 0 & 0 \\ 0 & \exp(i\frac{\pi}{4}) & 0 & 0 \\ 0 & 0 & \exp(i\frac{\pi}{4}) & 0 \\ 0 & 0 & 0 & \exp(-i\frac{\pi}{4}) \end{pmatrix},$$

$$\Rightarrow U = \exp(-i\frac{\pi}{4}). \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Le produit des matrices correspond au circuit suivant :



Sur le dessin l'état $|\psi\rangle$ entre par la gauche et la sortie est à droite
$(I_{2\times2} \otimes H)U(R_1 \otimes R_2)(I_{2\times2} \otimes H)|\psi\rangle$.
Calculons le produit :

$$R_1 \otimes R_2 = e^{-i\frac{\pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = e^{-i\frac{\pi}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

et

$$U(R_1 \otimes R_2) = e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

D'autre part

$$I_{2\times 2} \otimes H = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

et

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right),$$

puis

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right) \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) = \frac{1}{2} \left(\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right),$$

finalement on trouve

$$(I_{2\times 2} \otimes H)U(R_1 \otimes R_2)(I_{2\times 2} \otimes H) = e^{-i\frac{3\pi}{4}} \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right).$$

Cette matrice est égale à

$$\begin{aligned} & e^{-i\frac{3\pi}{4}} \begin{pmatrix} \sigma_x & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \\ &= e^{-i\frac{3\pi}{4}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \sigma_x - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{1} \right\} \\ &= e^{-i\frac{3\pi}{4}} \{ -|0\rangle\langle 0| \otimes \sigma_x + |1\rangle\langle 1| \otimes \mathbf{1} \}. \end{aligned}$$

Cette operation est une sorte de CNOT (mais par le CNOT standard).

Remarque : Pour obtenir la porte CNOT standard il faut utiliser des rotations avec un autre signe (c.a.d d'angle opposé) :

$$R_1 = \exp(+i\frac{\pi}{2}\frac{\sigma_1^2}{2}) \text{ et } R_2 = \exp(+i\frac{\pi}{2}\frac{\sigma_2^2}{2}).$$

On obtient alors (si on ne fait pas d'erreurs de signes !)

$$\begin{aligned}(I_{2\times2}\otimes H)U(R_1\otimes R_2)(I_{2\times2}\otimes H) &= e^{i\frac{3\pi}{4}}\{|0\rangle\langle 0|\otimes \mathbb{1} + |1\rangle\langle 1|\otimes\sigma_x\} \\ &= e^{i\frac{3\pi}{4}} \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)}_{\text{CNOT standard}}\end{aligned}$$