

Solution de la série 11
 Traitement Quantique de l'Information

Exercice 1 *Review of the Bloch sphere*

A general vector can be written in the form $\cos(\frac{\theta}{2}) |\uparrow\rangle + e^{i\phi} \sin(\frac{\theta}{2}) |\downarrow\rangle$ in the Bloch sphere.

- The bases vectors for the Z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$ which correspond to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$ respectively.
- The bases vectors for the Y are $\frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{i}{\sqrt{2}} |\downarrow\rangle$ and $\frac{1}{\sqrt{2}} |\uparrow\rangle - \frac{i}{\sqrt{2}} |\downarrow\rangle$ which correspond to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$.
- Bases vectors for the X basis are $\frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle$ and $\frac{1}{\sqrt{2}} |\uparrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\rangle$ corresponding to $(\theta = \frac{\pi}{2}, \phi = 0)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$ respectively.

The corresponding representation over the Bloch sphere is shown in Figure 1.

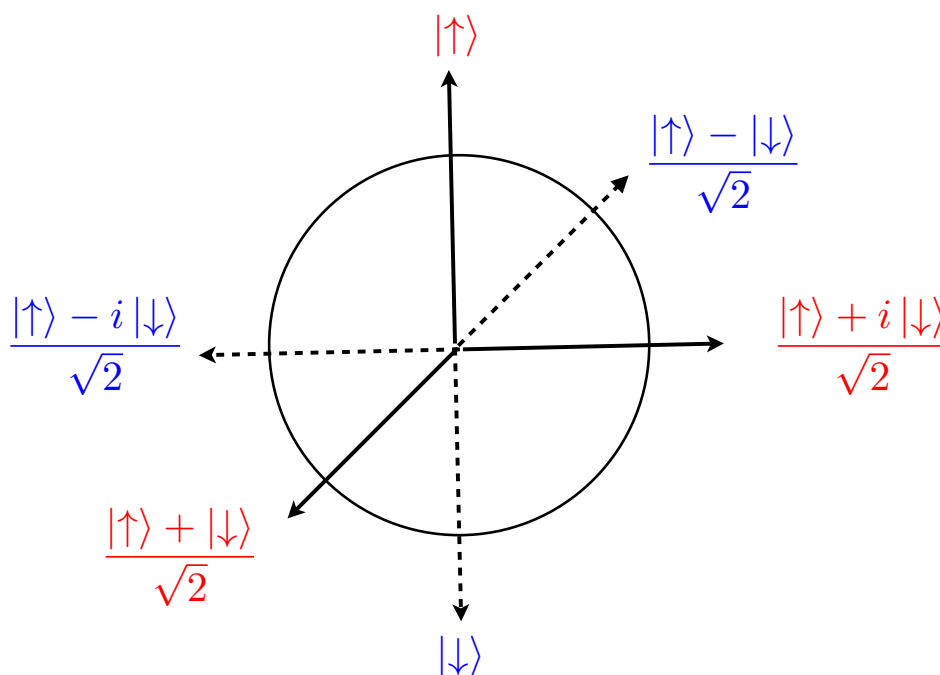


FIGURE 1 – Representation of basis vectors on Bloch Sphere

Using the general formula

$$\exp(i\frac{\theta}{2}(\vec{\sigma} \cdot \vec{n})) = \cos(\frac{\theta}{2})I + i\vec{\sigma} \cdot \vec{n}(\sin(\frac{\theta}{2})),$$

we obtain that

$$\begin{aligned}\exp(-i\frac{\alpha}{2}\sigma_x) &= \cos(\frac{\alpha}{2})I - i\sigma_x(\sin(\frac{\alpha}{2})) \\ &= \begin{pmatrix} \cos(\frac{\alpha}{2}) & -i\sin(\frac{\alpha}{2}) \\ -i\sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2}) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(-i\frac{\beta}{2}\sigma_y) &= \cos(\frac{\beta}{2})I - i\sigma_y(\sin(\frac{\beta}{2})) \\ &= \begin{pmatrix} \cos(\frac{\beta}{2}) & -\sin(\frac{\beta}{2}) \\ \sin(\frac{\beta}{2}) & \cos(\frac{\beta}{2}) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(-i\frac{\gamma}{2}\sigma_z) &= \cos(\frac{\gamma}{2})I - i\sigma_z(\sin(\frac{\gamma}{2})) \\ &= \begin{pmatrix} \cos(\frac{\gamma}{2}) - i\sin(\frac{\gamma}{2}) & 0 \\ 0 & \cos(\frac{\gamma}{2}) + i\sin(\frac{\gamma}{2}) \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}.\end{aligned}$$

The matrix $\exp(-i\frac{\alpha}{2}\sigma_x)$ is a rotation matrix of angle α around the X -axis, thus the state vector $\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})|\downarrow\rangle$ will transform to the vector $\cos(\frac{\theta+\alpha}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta+\alpha}{2})|\downarrow\rangle$. One can see that geometrically over the Bloch sphere, however one can also show by direct calculation that

$$\exp(-i\frac{\alpha}{2}\sigma_x) \left(\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})|\downarrow\rangle \right) = \cos(\frac{\theta+\alpha}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta+\alpha}{2})|\downarrow\rangle.$$

Similarly, one can see that $\exp(i\frac{\gamma}{2}\sigma_z)$ is a rotation of angle γ around the Z -axis. Therefore,

$$\exp(-i\frac{\gamma}{2}\sigma_z) \left(\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})|\downarrow\rangle \right) = e^{-i\frac{\gamma}{2}} \left(\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin(\frac{\theta}{2})|\downarrow\rangle \right).$$

Exercise 2 *Hamiltonian of the interaction of two spin $\frac{1}{2}$ particles*

- In the canonical bases, we have $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the tensor product rule one obtains that

$$\begin{aligned}\sigma_1^z \otimes \sigma_2^z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

thus the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} \hbar J & 0 & 0 & 0 \\ 0 & -\hbar J & 0 & 0 \\ 0 & 0 & -\hbar J & 0 \\ 0 & 0 & 0 & \hbar J \end{pmatrix}.$$

- In the bra-ket formalism one has $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, thus

$$\begin{aligned}\sigma_1^z \otimes \sigma_2^z &= (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \\ &= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.\end{aligned}$$

Therefore, we have

$$= \hbar J (|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|).$$

Notice that to verify this one can use

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which implies that

$$(|\uparrow\rangle\langle\uparrow|) \otimes (|\uparrow\rangle\langle\uparrow|) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly one can show that

$$|\uparrow\downarrow\rangle\langle\uparrow\downarrow| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|\downarrow\uparrow\rangle\langle\downarrow\uparrow| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|\downarrow\downarrow\rangle\langle\downarrow\downarrow| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- One can see that the eigen-values are $\hbar J$ corresponding to the eigen-vectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and $-\hbar J$ corresponding to the eigen-vectors $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$.

Exercice 3 *Identité utile pour la réalisation expérimentale de la porte CNOT par RMN*

- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ est diagonale donc

$$R_1 = R_2 = \begin{pmatrix} \exp(-i\frac{\pi}{4}) & 0 \\ 0 & \exp(i\frac{\pi}{4}) \end{pmatrix} = e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

- La porte de Hadamard est comme d'habitude $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

– Pour l’Hamiltonien on a :

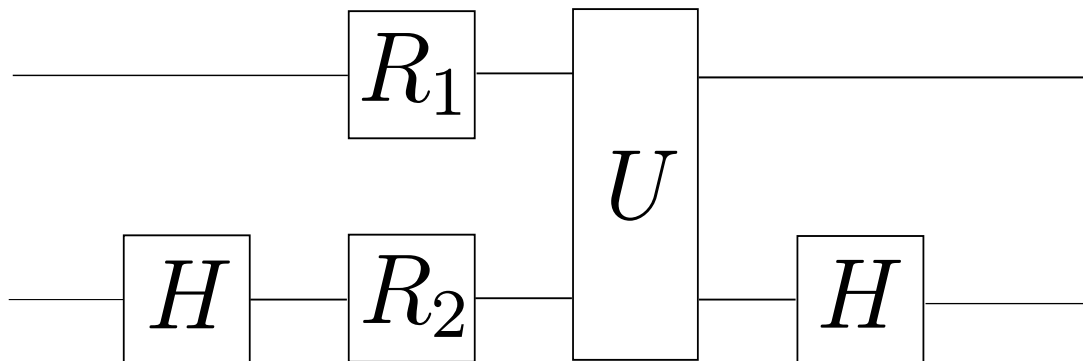
$$\mathcal{H} = \hbar J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

– Si on laisse évoluer pendant un temps $t = \frac{\pi}{4J}$ on trouve

$$U = \exp\left(-\frac{it}{\hbar}\mathcal{H}\right) = \exp\left(-\frac{i\pi}{4J\hbar}\mathcal{H}\right) = \begin{pmatrix} \exp(-i\frac{\pi}{4}) & 0 & 0 & 0 \\ 0 & \exp(i\frac{\pi}{4}) & 0 & 0 \\ 0 & 0 & \exp(i\frac{\pi}{4}) & 0 \\ 0 & 0 & 0 & \exp(-i\frac{\pi}{4}) \end{pmatrix},$$

$$\Rightarrow U = \exp\left(-i\frac{\pi}{4}\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Le produit des matrices correspond au circuit suivant :



Sur le dessin l’état $|\psi\rangle$ entre par la gauche et la sortie est à droite $(I_{2 \times 2} \otimes H)U(R_1 \otimes R_2)(I_{2 \times 2} \otimes H)|\psi\rangle$.

Calculons le produit :

$$\begin{aligned} R_1 \otimes R_2 &= e^{-i\frac{\pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\ &= e^{-i\frac{\pi}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \end{aligned}$$

et

$$\begin{aligned}
U(R_1 \otimes R_2) &= e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}$$

D'autre part

$$I_{2 \times 2} \otimes H = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right)$$

et

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{array} \right),$$

puis

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{array} \right) \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right) = \frac{1}{2} \left(\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{array} \right),$$

finalement on trouve

$$(I_{2 \times 2} \otimes H)U(R_1 \otimes R_2)(I_{2 \times 2} \otimes H) = e^{-i\frac{3\pi}{4}} \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right).$$

Cette matrice est une "sorte de porte CNOT". Elle est égale à

$$\begin{aligned}
&e^{-i\frac{3\pi}{4}} \begin{pmatrix} \sigma_x & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \\
&= e^{-i\frac{3\pi}{4}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \sigma_x - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{1} \right\} \\
&= e^{-i\frac{3\pi}{4}} \{ |0\rangle\langle 0| \otimes \sigma_x - |1\rangle\langle 1| \otimes \mathbf{1} \}.
\end{aligned}$$

Cette operation flip un bit si le bit de controle est dans l'état $|0\rangle$ et change la phase du bit si le bit de controle est dans l'état $|1\rangle$.

Remarque : Pour obtenir la porte CNOT standard il faut utiliser des rotations avec un autre signe (c.a.d d'angle opposé) :

$$R_1 = \exp(+i\frac{\pi}{2}\frac{\sigma_1^2}{2}) \text{ et } R_2 = \exp(+i\frac{\pi}{2}\frac{\sigma_2^2}{2}).$$

On obtient alors (si on ne fait pas d'erreurs de signes!)

$$\begin{aligned} (I_{2 \times 2} \otimes H)U(R_1 \otimes R_2)(I_{2 \times 2} \otimes H) &= e^{i\frac{3\pi}{4}} \{ |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x \} \\ &= e^{i\frac{3\pi}{4}} \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right)}_{\text{CNOT standard}} \end{aligned}$$