**Homework December 2, 2009. Quantum information theory and computation**

**Problem 1. Source coding of mixed states: a conjecture.**

We consider a source of mixed states $\rho_x$ occuring each with probabilities $p_x$. Messages are $N$ letter strings of the form $\rho_{x_1} \otimes .... \otimes \rho_{x_2}$ and have a probability $p_{x_1}...p_{x_N}$. In this exercise we want to give some support to the conjecture that the achievable rate of compression for a source of mixed states is equal to the Holevo quantity

$$\chi(\{p_x, \rho_x\}) = S(\rho) - \sum_x p_x S(\rho_x), \qquad \rho = \sum_x p_x \rho_x$$

Note that in the case of a source of pure states $\rho_x = |\phi_x\rangle\langle\phi_x|$ the Holevo quantity $\chi(\rho)$ reduce to $S(\rho)$ which is the optimal achievable rate given by Schumacher's theorem.

a) Take a source constituted of the unique letter $\rho_0$ occuring with probability $p_0 = 1$. How many bits are needed to compress this source ? What is the value of $\chi(\rho)$ ? Is this consistent ?

b) Now consider a source of mixed mutually orthogonal states. Two mixed states are said to be mutually orthogonal if

$$Tr \rho_x \rho_y = 0, \qquad x \neq y$$

Construct purifications $|\Psi_x\rangle$ of $\rho_x$ that satisfy (*hint: use the spectral decomposition*)

$$\langle\Psi_x|\Psi_y\rangle = 0, \qquad x \neq y$$

What would be an encoding scheme achieving a compression rate of $H(X) = -\sum_x p_x \log p_x$ ? Why would this rate be optimal ? Check that in the present case we have

$$H(X) = \chi(\rho)$$

*hint: no big calculations.*

**Problem 2. Product state capacity of the quantum depolarizing channel.**

Consider the map from $2 \times 2$ density matrices to $2 \times 2$ density matrices

$$\mathcal{Q}(\rho) = (1 - \epsilon)\rho + \frac{\epsilon}{2} I$$

and define

$$C(\epsilon) = \max_{p_x,\rho_x} \left\{ S(\sum_x p_x \mathcal{Q}(\rho_x)) - \sum_x p_x S(\mathcal{Q}(\rho_x)) \right\}$$

where the supremum is taken over a finite alphabet $\{\rho_x\}$ of $2 \times 2$ density matrices and a probability distribution over this alphabet. We will accept (and there is a proof) that the max is attained for pure states $\rho_x = |\phi_x\rangle\langle\phi_x|$ (not necessarily orthogonal).

The map above is a model for a quantum channel, aclled the depolarizing channel. This is a channel were the output density matrix stays intact with probability $1 - \epsilon$ and becomes completely random with probability $\epsilon$. The quantity $C(\epsilon)$ is a formula for the capacity, i.e the maximal rate of transmission, for classical messages encoded in tensor products of quantum states (say polarization states of photons to be sent over an optic fiber). This capacity formula was demonstrated by Holevo-Schumacher-Westmoreland. We see that the Holevo quantity associated to $\{p_x, \mathcal{Q}(\rho_x)\}$ plays the role of a mutual information $I(X;Y)$. Analogously with the classical case we have to maximize over all possible input distributions in order to find the channel capacity.

a) Prove that

$$C(\epsilon) = \ln 2 - H\big(\frac{\epsilon}{2}\big)$$

were $H(x)$ is the usual binary entropy function (defined with natural log).

*hint: no big optimization.*