Homework 2. Quantum information theory and computation - Winter semester 2011

Problem 1. Polarization measurements and uncertainty relation.

Photons that pass through a polarizer at an angle θ are prepared in the state $|\theta\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle$. A measurement apparatus consists of an analyzer at an angle α and a detector. measurements results are registered in a random variable $p_\alpha = \pm 1$. When the detector clicks, the photon has been observed in state $|\alpha\rangle = \cos\alpha|x\rangle + \sin\alpha|y\rangle$ and we set $p_\alpha = +1$. When it does not click the photon has been observed in the state $|\alpha_\perp\rangle$ ($\alpha_\perp = \alpha + \frac{\pi}{2}$ and we register $p_\alpha = -1$.

a) Derive the probabilities of detection and non detection, $Prob(p_\alpha = \pm 1)$ form the Born rule (measurement postulate). Then compute the expectation and variance of p_α. Fix θ and observe how they vary as a function of α.

b) Consider now the "observable" defined as $P_\alpha = (+1)|\alpha\rangle\langle\alpha| + (-1)|\alpha_\perp\rangle\langle\alpha_\perp|$. Check that

$$\langle P_\alpha\rangle \equiv \langle\theta|P_\alpha|\theta\rangle, \qquad (\Delta P_\alpha)^2 \equiv \langle\theta|P_\alpha^2|\theta\rangle - \langle\theta|P_\alpha|\theta\rangle^2$$

agree with the results of a).

c) Consider two angles α and β and compute the commutator $[P_\alpha, P_\beta] = P_\alpha P_\beta - P_\beta P_\alpha$. Check (say by fixing α and β and plotting as a function of θ) that Heisenberg's uncertainty principle is satisfied for any $|\theta\rangle$, namely

$$\Delta P_\alpha \Delta P_\beta \geq \frac{1}{2}|\langle\theta|[P_\alpha, P_\beta]|\theta\rangle|.$$

Remark: you can write the matrices corresponding to P_α and P_β in the computational basis to see how they look like. But the above calculations are more easily done directly in Dirac notation instead of matrix form.

Problem 2. Heisenberg uncertainty relation

a) Prove Heisenberg's uncertainty relation (see notes)

$$\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|.$$

Hint: Express the positivity of the variance of the observable $A' + \lambda B'$ (λ areal number) for of A' and B' where $A' = A - \langle\psi|A|\psi\rangle$ and similarly for B. Use Cauchy-Schwarz.

b) Take $|\psi\rangle = |0\rangle$, $A = X$, $B = Y$ and apply the inequality. Here X, Y, Z are the three Pauli matrices defined in the notes.

c) This question lies a bit outside of the scope of this course but anyone learning QM should be exposed to it. Consider now the Hilbert space $\mathcal{H} = L^2(\mathbf{R})$ of a particle in one dimensional space. The states are wave functions $\psi(x)$ that are square integrable. The position observable is the multiplication operator $\hat{x}$ defined by $(\hat{x}\psi)(x) = x\psi(x)$ and the momentum operator $\hat{p}$ defined by $(\hat{p}\psi)(x) = -i\hbar\frac{d}{dx}\psi(x)$. Compute the commutator $[\hat{x}, \hat{p}]$ and interpret the uncertainty relation.

Problem 3. Entropic uncertainty principle

Let A and B be two observables with non-degenerate eigenvector basis $\{|a\rangle\}$ and $\{|b\rangle\}$. Consider the two (classical) probability distributions given by the measurement postulate when the system is in state $|\psi\rangle$. Each probability distribution has a corresponding (classical) Shannon entropy, call them H_A and H_B. Prove the "entropic uncertainty principle" mentioned in the notes:

$$H_A + H_B \geq -2\log\left(\frac{1 + \max|\langle a|b\rangle|}{2}\right).$$

Hint: Reason geometrically to show that $|\langle a|\psi\rangle\langle\psi|b\rangle|^2 \leq |\langle a|b\rangle|^2$.