Exercises september 21, 2011. Quantum Information Theory and Computation.

Problem 1. Mach-Zehnder interferometer.

A beam splitter (see figure) is a semi transparent mirror which separates a beam of light in two equal intensity beams. Here is a simple quantum mechanical model of this device. Suppose each photon lives in a two dimensional Hilbert space spanned by the basis states $|T\rangle$, $|R\rangle$ corresponding to the transmitted and refected beam. When a photon in the state $|T\rangle$ (see picture) hits the beam splitter the electrons of the material absorb it and reemit it in the the new state $\frac{1}{\sqrt{2}}|T\rangle + \frac{1}{\sqrt{2}}|R\rangle$. We know that this purely dynamical process is modeled by a unitary "time evolution" or "transition matrix".

a) Write down the unitary matrix in the basis $|T\rangle$, $|R\rangle$. In QIT this matrix is called a Hadamard gate and is denoted by H.

b) Take a general incoming state $|\psi\rangle = \alpha|T\rangle+\beta|R\rangle$ and compute the outgoing state. For incoming light where all photons are in the state $|\psi\rangle$ what are the intensities of the outgoing ligth beams ?

Consider the following setup (Mach-Zehnder interferometer). An incoming *horitontal* beam is splitted in two equal intensity beams which are then recollected thanks to perfectly reflecting mirrors and splitted again in two equal intensity beams (here we suppos the mirrors do not affect the states of the photons). Two detectors D_A and D_B click each time a photon hits them.

c) If you reason "classicaly" what is the probability that D_A clicks ? And that of D_B ?

d) Now do the quantum mechanical computation. One way to proceed is to first find out the unitary transition matrix between the incoming state and the outgoing state.

e) Now take another kind of beam splitters such that $|T\rangle \to \frac{1}{\sqrt{2}}|T\rangle + \frac{i}{\sqrt{2}}|R\rangle$ (find the other transition) and compute again the probabilities of hearing clicks at D_A and D_B.

f) Now we introduce a "phase shifter" on the upper arm of the interferometer. This is a unitary device $S|R\rangle = e^{i\Phi}|R\rangle$, $S|T\rangle = |T\rangle$. Compute the intensities measured by D_A and D_B.

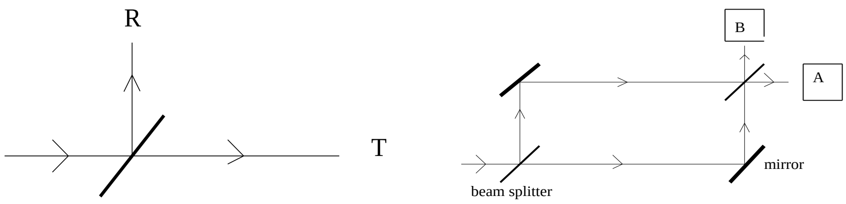

Figure 1: beam splitter and Mach-Zehnder interferometer

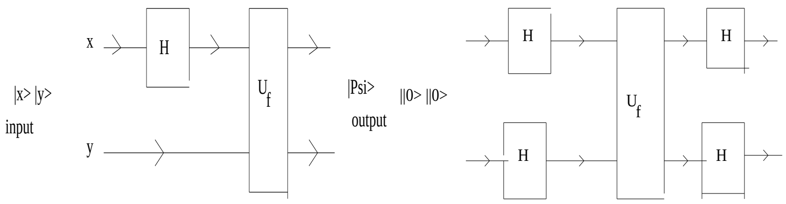

Figure 2: Deutsch illustration of quantum parallelism

Problem 2. Quantum parallelism.

Suppose we want to compute all possible values of the map $f : \{0, 1\} \to \{0, 1\}$ using a quantum unitary evolution. The idea is to store the argument of f in some Qbit $|x\rangle$ ($x = 0, 1$) and the result in another Qbit $|y\rangle$. Our Hilbert space is spanned by the four basis states $\{|x\rangle \otimes |y\rangle\}$ where $\otimes$ means the tensor product (or kronecker product). Prove that

$$U_f|x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus f(x)\rangle$$

is a unitary map of the four dimensional Hilbert space to itself. Consider the quantum circuit where $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate acting on the first Qbit.

a) What does this circuit do to the input state $|0\rangle \otimes |0\rangle$?

b) Suppose we have a way to do a measurement. What is the probability of observing $f(0)$ and $f(1)$.

The following is a problem first posed by David Deutsch. We want to determine if the function f is constant, $f(0) = f(1)$, or not. Classicaly the only way to do that is to evaluate $f(0)$ and $f(1)$ and observe each value separately: two evaluations are required. Quantum mechanicaly only one evaluation is needed. Find out why, by looking at the following quantum operation of figure .