ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 4 Information Theory and Coding
Homework 2 September 27,2011

PROBLEM 1. A source has an alphabet of 4 letters. The probabilities of the letters and two possible sets of binary code words for the source are given below:

Letter	Prob.	Code I	Code II
a_1	0.4	1	1
a_2	0.3	01	10
a_3	0.2	001	100
a_4	0.1	000	1000

For *each* code, answer the following questions (no proofs or numerical answers are required).

(a) Is the code instantaneous?

(b) Is the code uniquely decodable?

(c) What is the mutual information betweem the event that the source letter is a_1 and the event that the first letter of the codeword is 1? (For mutual information between events, consider the mutual information between corresponding indicator random variables—a random variable which is 1 if the event takes place 0 otherwise.)

(d) Give an heuristic description of the purpose of the first letter in the code words of code II.

PROBLEM 2. Let $\bar{M} = \sum_{i=1} p_i \log(l_i)$ be the expected value of the logarithm of the code word lengths l_i associated with an encoding of a random variable X with distribution p. Let $\bar{M}_1 = \min \bar{M}$ over all instantaneous codes; and let $\bar{M}_2 = \min \bar{M}$ over all uniquely decodable codes. What inequality relationship exists between $\bar{M}_1$ and $\bar{M}_2$?

PROBLEM 3. Consider the following method for constructing binary code words for a random variable U which takes values $\{a_1, \ldots, a_m\}$ with probabilities $P(a_1), \ldots, P(a_m)$. Assume that $P(a_1) \geq P(a_2) \geq \cdots \geq P(a_m)$. Define

$$Q_i = \sum_{k=1}^{i-1} P(a_k) \quad \text{for } i > 1;\ Q_1 = 0.$$

The code word assigned to the message a_i is formed by finding the binary expansion of $Q_i < 1$ (i.e, $1/2 = 100\ldots$, $1/4 = 0100\ldots$, $5/8 = 1010\ldots$) and then truncating this expansion to the first l_i bits where $l_i = \lceil -\log_2 P(a_i) \rceil$.

(a) Construct binary code words for the probability distribution $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$.

(b) Prove that the method described above yields an instantaneous code (i.e., no codeword is a prefix of another) and the average codeword length $\bar{L}$ satisfies

$$H(X) \leq \bar{L} < H(X) + 1.$$

Problem 4. A random variable takes values on an alphabet of K letters, and each letter has the same probability. These letters are encoded into binary words so as to minimize the average code word length. Define j and x so that $K = x2^j$, where j is an integer and $1 \leq x < 2$.

(a) Do any code words have lengths not equal to j or $j + 1$? Why?

(b) In terms of j and x, how many code words have length j?

(c) What is the average code word length?

Problem 5.

(a) A source has an alphabet of 4 letters, a_1, a_2, a_3, a_4, and we have the condition $P(a_1) > P(a_2) = P(a_3) = P(a_4)$. Find the smallest number q such that $P(a_1) > q$ implies that $n_1 = 1$ where n_1 throughout this problem is the length of the codeword for a_1 in a Huffman code.

(b) Show by example that if $P(a_1) = q$ (your answer in part (a)), then a Huffman code exists with $n_1 > 1$.

(c) Now assume the more general condition, $P(a_1) > P(a_2) \geq P(a_3) \geq P(a_4)$. Does $P(a_1) > q$ still imply that $n_1 = 1$? Why or why not?

(d) Now assume that the source has an arbitrary number K of letters with $P(a_1) > P(a_2) \geq \cdots \geq P(a_K)$. Does $P(a_1) > q$ now imply $n_1 = 1$?

(e) Assume $P(a_1) \geq P(a_2) \geq \cdots \geq P(a_K)$. Find the largest number q' such that $P(a_1) < q'$ implies that $n_1 > 1$.

Problem 6. Let X be a random variable taking values in M points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) H(Y)$$

where Y is a random variable taking values in $M - 1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1 - \alpha)$; $1 \leq j \leq M - 1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)$$

and determine the condition for equality.