## ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 18 Homework 8 Information Theory and Coding November 15, 2011

PROBLEM 1. Channels with memory have higher capacity. Consider a binary symmetric channel with  $Y_i = X_i \oplus Z_i$ , where  $\oplus$  is mod 2 addition, and  $X_i, Y_i \in \{0, 1\}$ .

Suppose that  $\{Z_i\}$  has constant marginal probabilities  $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$ , but that  $Z_1, Z_2, \ldots, Z_n$  are not independent. Assume that  $(Z_1, \ldots, Z_n)$  is independent of the input  $(X_1, \ldots, X_n)$ . Let C = 1 - H(p, 1 - p). Show that

$$\max_{p_{X_1,X_2,...,X_n}} I(X_1,X_2,...,X_n;Y_1,Y_2,...,Y_n) > nC.$$

PROBLEM 2. Consider two discrete memoryless channels. The input alphabet, output alphabet, transition probabities and capacity of the k'th channel is given by  $\mathcal{X}_k$ ,  $\mathcal{Y}_k$ ,  $p_k$  and  $C_k$  respectively (k = 1, 2). The channels operate independently. A communication system has access to both channels, that is, the effective channel between the transmitter and receiver has input alphabet  $\mathcal{X}_1 \times \mathcal{X}_2$ , output alphabet  $\mathcal{Y}_1 \times \mathcal{Y}_2$  and transition probabilities  $p_1(y_1|x_1)p_2(y_2|x_2)$ . Find the capacity of this channel.

Problem 3. Show that a cascade of n identical binary symmetric channels,

$$X_0 \to \boxed{\mathrm{BSC} \ \#1} \to X_1 \to \cdots \to X_{n-1} \to \boxed{\mathrm{BSC} \ \#n} \to X_n$$

each with raw error probability p, is equivalent to a single BSC with error probability  $\frac{1}{2}(1-(1-2p)^n)$  and hence that  $\lim_{n\to\infty} I(X_0;X_n)=0$  if  $p\neq 0,1$ . Thus, if no processing is allowed at the intermediate terminals, the capacity of the cascade tends to zero.

PROBLEM 4. Consider a memoryless channel with transition probability matrix  $P_{Y|X}(y|x)$ , with  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ . For a distribution Q over  $\mathcal{X}$ , let I(Q) denote the mutual information between the input and the output of the channel when the input distribution is Q. Show that for any two distributions Q and Q' over  $\mathcal{X}$ ,

(a)
$$I(Q') \le \sum_{x \in \mathcal{X}} Q'(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x') Q(x')} \right)$$

(b)
$$C \le \max_{x} \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x') Q(x')} \right)$$

where C is the capacity of the channel. Notice that this upper bound to the capacity is independent of the maximizing distribution.