ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Principles of Digital Communications: Assignment date: Feb 22, 2012
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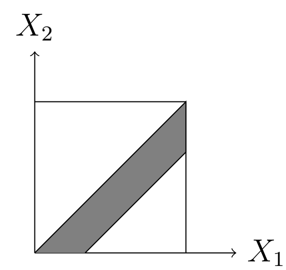
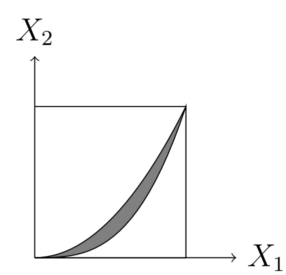
Solution of Homework 1

Problem 1. *(Probabilities of Basic Events)*
In each case, the shaded region represents the (X_1, X_2) values satisfying the corresponding inequalities. Since X_1 and X_2 are independent and uniformly distributed, the area of the shaded region gives the probability of the inequality being satisfied. We use $\mathbb{P}[\mathcal{A}]$ to denote the probability of an event $\mathcal{A}$.

1.
$$\mathbb{P}\left[0 \le X_1 - X_2 \le \frac{1}{3}\right] = \frac{1}{2} - \frac{1}{2} \times (\frac{2}{3} \times \frac{2}{3}) = \frac{5}{18}.$$

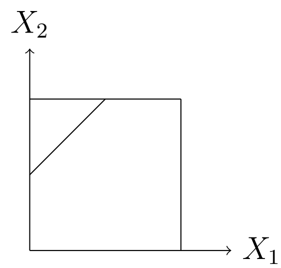


2.
$$\mathbb{P}\left[X_1^3 \le X_2 \le X_1^2\right] = \int_0^1 (X_1^2 - X_1^3) dX_1 = \left[\frac{X_1^3}{3} - \frac{X_1^4}{4}\right]_0^1 = \frac{1}{12}.$$



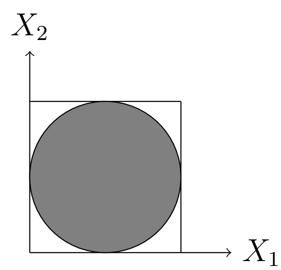
3.

$$\mathbb{P}\left[X_2 - X_1 = \frac{1}{2}\right] = 0.$$



4.

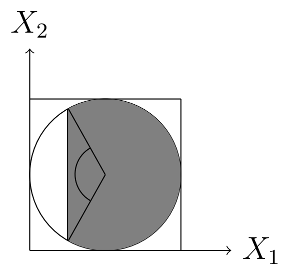
$$\mathbb{P}\left[(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \le (\frac{1}{2})^2\right] = \pi(\frac{1}{2})^2 = \frac{\pi}{4}.$$



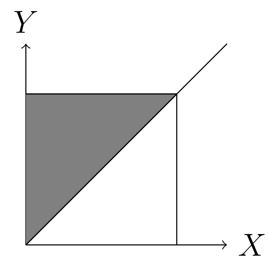
5. In this part we have

$$\begin{aligned}\mathbb{P}&\left[(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \le (\frac{1}{2})^2 \mid X_1 \ge \frac{1}{4}\right] \\ &= \frac{\mathbb{P}\left[(X_1 - \frac{1}{2})^2 + (X_2 - \frac{1}{2})^2 \le (\frac{1}{2})^2\ ,\ X_1 \ge \frac{1}{4}\right]}{\mathbb{P}\left[X_1 \ge \frac{1}{4}\right]} \\ &= \frac{\frac{\pi}{6} + \frac{\sqrt{3}}{16}}{\frac{3}{4}}\end{aligned}$$

It can easily be seen that probability term in the numerator is equal to the area of the shaded region in the figure below. We can divide the shaded area to two parts, triangular and sub circular. It is easy to show that the angle of the triangle on the picture is 120° so the sub circular part consists of $\frac{2}{3}$ of the circle area. So sub circular part area is $\frac{2}{3}\ \pi(\frac{1}{2})^2 = \frac{\pi}{6}$ and triangular part area is $\frac{\sqrt{3}}{16}$ and summing the area of these two parts, we reach to the final result.



Problem 2. *(Conditional Distribution)* Probability mass has been distributed uniformly on the upper triangular area according to the shape below:



1. If X and Y were independent then the distribution of X would not depend on Y. This is clearly not the case. In fact, the range of values taken by X is between 0 and Y.

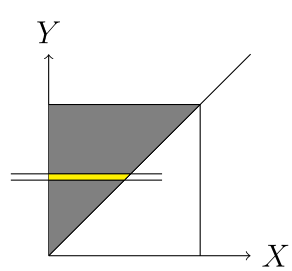
2. The integral of $f_{X,Y}(x, y)$ must be 1. Hence $A \times \frac{1}{2} = 1$ and so $A = 2$.

3. We know that $f_Y(y)\ dy = \mathbb{P}\left[y < Y < y + \ dy\right]$ but for a special y as can be seen from the figure below this probability mass is equal to A times the area of a rectangle with length y and width dy when $0 \le y \le 1$.

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & otherwise \end{cases}$$

   Or more formally

$$f_Y(y) = \int_0^1 f_{X,Y}(x, y)\ dx = \int_0^y 2\ dx = 2\ y$$



4. If we take $y < Y < y + dy$ then as can be seen from the previous part the values random variable X can take are uniformly distributed between 0 and y and so $\Phi(y) = \mathbb{E}[X|Y = y] = \frac{y}{2}$.

5. $\Phi(y)$ is a function of Y so it is a random variable and we can compute its expectation value. $\mathbb{E}[\Phi(Y)] = \int_0^1 \Phi(y) f_Y(y)\ dy = \frac{1}{3}$.

6. We compute the $\mathbb{E}[X]$ using the definition. $\mathbb{E}[X] = \int\int x f_{X,Y}(x,y)\ dx\ dy = \int_0^1 [\int_0^y 2x\ dx]\ dy = \frac{1}{3}$ and it is seen that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$. This result also holds in general.

Problem 3. *(Basic Probabilities)*

a) First, we find the probability of the complement of the event, namely the probability of drawing only black balls. This probability is equal to

$$\mathbb{P}[\text{All } k \text{ balls are black}] = \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

Therefore the probability of drawing at least one white ball is equal to

$$\mathbb{P}[\text{At least one ball is white}] = 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

b) Define the following random variables

$$X = \begin{cases} 0 & \text{If the chosen coin is fair,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$Y = \begin{cases} 00 & \text{If both outcomes are tail,} \\ 01 & \text{If the first one is tail, the second one is head,} \\ 10 & \text{If the first one is head, the second one is tail,} \\ 11 & \text{If both outcomes are head.} \end{cases}$$

So having these two random variables defined, we want to compute $\mathbb{P}[X = 0|Y = 11]$. So we can write

$$\begin{aligned} \mathbb{P}[X = 0|Y = 11] &= \frac{\mathbb{P}[X = 0, Y = 11]}{\mathbb{P}[Y = 11]} \\ &= \frac{1/2 \times 1/4}{\mathbb{P}[Y = 11]} \\ &= \frac{1/8}{\mathbb{P}[Y = 11]}. \end{aligned}$$

Then for $\mathbb{P}[Y = 11]$ we have

$$\begin{aligned}\mathbb{P}[Y = 11] &= \mathbb{P}[X = 0] \cdot \mathbb{P}[Y = 11|X = 0] + \mathbb{P}[X = 1] \cdot \mathbb{P}[Y = 11|X = 1] \\ &= 1/2 \times 1/4 + 1/2 \times 1 \\ &= 5/8.\end{aligned}$$

So, finally we have

$$\mathbb{P}[X = 0|Y = 11] = \frac{1/8}{5/8} = 1/5.$$

Problem 4. *(Basic Probabilities)*
T is a random variable and can take discrete values and we have

1. $\mathbb{P}[T = n] = \mathbb{P}[$in the first $n-1$ tosses we obtain tail and obtain head in the last toss $]$ and so $\mathbb{P}[T = n] = p(1-p)^{n-1} \quad n = 1, 2, \cdots$.

2. $\{T > k\}$ is an event and if $n > k$ then $\{T = n\} \subset \{T > k\}$ so we can write:

$$\mathbb{P}[T = n|T > k] = \frac{\mathbb{P}[T = n, T > k]}{\mathbb{P}[T > k]} = \frac{\mathbb{P}[T = n]}{\mathbb{P}[T > k]} = \frac{p(1-p)^{n-1}}{(1-p)^k} = p(1-p)^{n-k-1}$$

   When we have used the fact that $\mathbb{P}[T > k]$ is the probability that the outcome of the first k tosses is tail which is equal to $(1-p)^k$. Hence $\mathbb{P}[T = n|T > k] = \mathbb{P}[T = n-k]$. This makes sense. Due to the independence of each toss, the probability that $T = n$ given that the first k tosses $(k < n)$ lead to tail is the probability that in the remaining tosses we obtain $n-k-1$ tails followed by a head.

3. After we have observed k tails, the expectation value of T is k + expected number of tosses we still will do. Since the process is memoryless(in the sense demonstrated in the previous question) the latter is again $\mathbb{E}[T]$.