ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Principles of Digital Communications: Assignment date: May 16, 2012
Summer Semester 2012 Due date: May 23, 2012

Homework 12

Reading Part for next Wednesday: Entire Chapter 7 (Complex-Valued Random Variables and Processes) without the appendices. (8p) (Appendices browsing through as needed).

Problem 1. Here are final rules for your project. You will be given a text of length 100. It is in plain English with no special characters. The two laptops will be placed 5 meters apart. Of course WiFi, Bluetooth, etc should be switched off. :-)

To start you can press on both computers a key. After that, all processing should be done automatically. On the screen of the receiver print out the text. On the day of the project presentation, send us a zip file containing

(i) All programs you use with instructions on how to install them and what hardware is required to run them.

(ii) A write up between 2 and 4 pages which describes the system, explains your choices of parameters; include in there also a paragraph which comments on what parts of the system could be improved.

Evaluation:

1. You will receive maximum points if (i) your system is "reliable" and (ii) it transmits at a "reasonable" speed; "reliable" means that at most 2 of the 100 characters are wrong; "reasonable speed" means no slower than 1/10th the speed of the fastest system among all the projects.

2. The fastest system gets bonus points. Further, we will give bonus points for the most sophisticated system (the system which has the most features) and perhaps some other categories.

Problem 2. *(Convolutional Code)*
The following equations define a convolutional code for a data sequence $d_i \in \{-1, 1\}$:

$$x_{3n} = d_{2n} \cdot d_{2n-1} \cdot d_{2n-2} \tag{1}$$
$$x_{3n+1} = d_{2n+1} \cdot d_{2n-2} \tag{2}$$
$$x_{3n+2} = d_{2n+1} \cdot d_{2n} \cdot d_{2n-2} \tag{3}$$

1. Draw an implementation of the encoder of this convolutional code, using only delay elements D and multipliers. *Hint:* Split the data sequence d into two sequences, one containing only the even-indexed samples, the other containing only the odd-indexed samples.

2. What is the rate of this convolutional code?

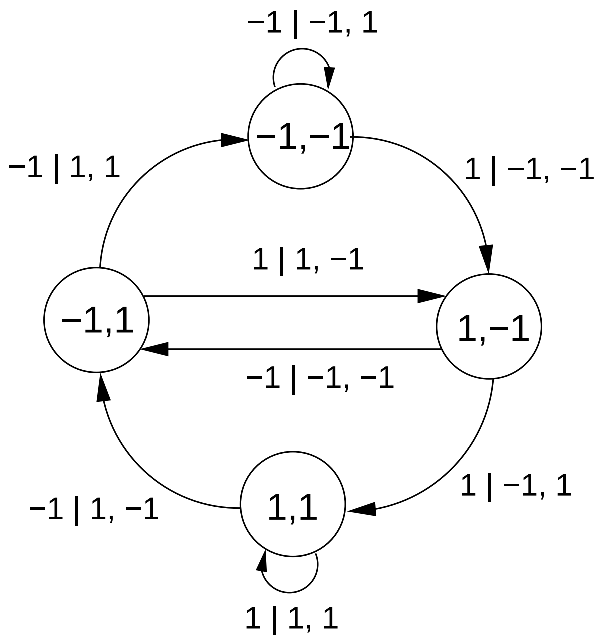
3. Draw the state diagram for this convolutional encoder.

4. Does the formula for the upper bound on P_b that was derived in class still hold? If not, make the appropriate changes.

5. (optional) Now suppose that the code is used on an AWGN channel. The energy available per source digit is E_b and the power spectral density of the noise is $N_0/2$. Give the detour flowgraph, and derive an upper bound on the bit error probability P_b as a function of E_b/N_0.

Problem 3. *(Convolutional Encoder, Decoder and Error Probability)*
In this problem we send a sequence $\{D_j\}$ taking values in $\{-1, +1\}$, for $j = 0, 1, 2, \cdots, k-1$, using a convolutional encoder. The channel adds white Gaussian noise to the transmitted signal. If we let X_j denote the transmitted value, then, the received value is: $Y_j = X_j + Z_j$, where $\{Z_j\}$ is a sequence of i.i.d. zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. The receiver implements the Viterbi algorithm.

1. Convolutional Encoder is described by the finite state machine depicted below. The transitions are labeled by $D_j | X_{2j}, X_{2j+1}$, and the states by (D_{j-1}, D_{j-2}). We assume that the initial state is $(1, 1)$.



(a) What is the rate of this encoder?

(b) Sketch the block diagram of the encoder that corresponds to this finite state machine. How many shift registers do you need?

(c) Draw a section of the trellis representing this encoder.

2. Let X_j^i denote the output of the convolutional encoder at time j when we transmit hypothesis i, $i = 0, \cdots, m-1$, where m is the number of different hypotheses. Assume that the received vector is $\bar{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = (-1, -3, -2, 0, 2, 3)$. It is the task of the receiver to decide which hypothesis i was chosen or, equivalently, which vector $\bar{X}^i = (X_1^i, X_2^i, X_3^i, X_4^i, X_5^i, X_6^i)$ was transmitted.

   (a) Use the Viterbi algorithm to find the most probable transmitted vector $\bar{X}^i$.

3. For the performance analysis of the code:

   (a) Suppose that this code is decoded using the Viterbi algorithm. Draw the detour flow graph, and label the edges by the input weight using the symbol I, and the output weight using the symbol D.

   (b) Considering the following generating function

$$T(I, D) = \frac{ID^4}{1 - 3ID},$$

   What is the value of

$$\sum_{i,d} ia(i,d)e^{-\frac{d}{2N_0}},$$

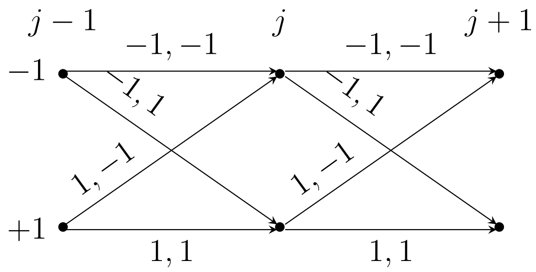
where $a(i,d)$ is the number of detours with i bit errors and d channel errors? First compute this expression, then give an interpretation in terms of probability of error of this quantity.

Hints: Recall that the generating function is defined as $T(I,D) = \sum_{i,d} a(i,d) D^d I^i$. *You may also use the formula* $\sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2}$ *if* $|q| < 1$.

Problem 4. *(Trellis with Antipodal Signals)*
Assume that the sequence $X_1, X_2, \ldots$ is sent over an additive white Gaussian noise channel, *i.e.*,

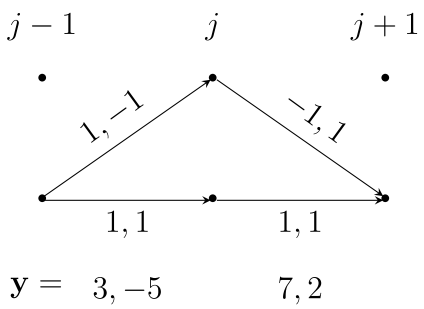
$$Y_i = X_i + Z_i,$$

where the Z_i are i.i.d. zero-mean Gaussian random variables with variance σ^2. The sequence X_i is the output of a convolutional encoder described by the following trellis.

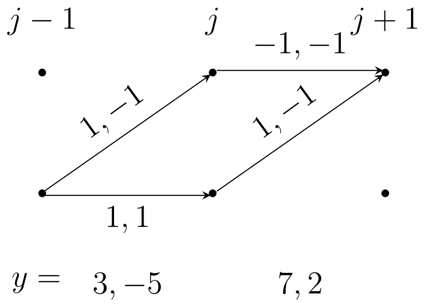


As the figure shows, the trellis has two states labeled with $+1$ and -1, respectively. The probability assigned to each of the two branches leaving any given state is $\frac{1}{2}$. The trellis is also labeled with the output produced when a branch is traversed and with the trellis depths $j-1$, j, $j+1$.

1. Consider the two paths in the following picture. Which of the two paths is more likely if the corresponding channel output subsequence $y_{2j-1}, y_{2j}, y_{2j+1}, y_{2(j+1)}$ is $3, -5, 7, 2$?

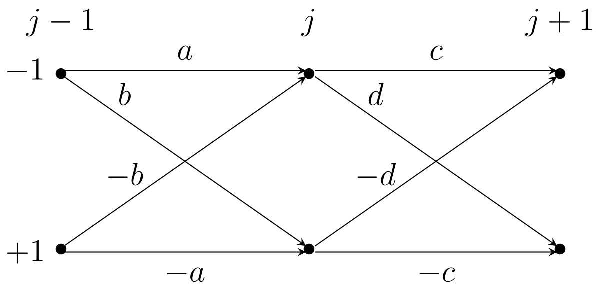


2. Now, consider the following two paths with the same channel output as in the previous question. Find again the most likely of the two paths.



3. If you have made no mistake in the previous two questions, the state at depth j of the most likely paths is the same in both cases. This is no coincidence as we will now prove.

   The first step is to remark that the metric has to be as in the following picture for some value of a, b, c, and d.



   (a) Now let us denote by $\sigma_k \in \{\pm 1\}$ the state at depth k, $k = 0, 1, \cdots$, of the maximum likelihood path. Assume that a genie tells you that $\sigma_{j-1} = 1$ and $\sigma_{j+1} = 1$. In terms of a, b, c, d, write down a necessary condition for $\sigma_j = 1$. (The condition is also sufficient up to ties.)

   (b) Now assume that $\sigma_{j-1} = 1$ and $\sigma_{j+1} = -1$. What is the condition for choosing $\sigma_j = 1$?

   (c) Now assume that $\sigma_{j-1} = -1$ and $\sigma_{j+1} = 1$. What is the condition for $\sigma_j = 1$?

   (d) Now assume that $\sigma_{j-1} = -1$ and $\sigma_{j+1} = -1$. What is the condition for $\sigma_j = 1$?

   (e) Are the four conditions equivalent? Justify your answer.

   (f) Comment on the advantage, if any, implied by your answer to the previous question.