ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Principles of Digital Communications: Summer Semester 2012 | Assignment date: Mar 28, 2012 | Due date: Apr 4, 2012

Homework 7

Reading Part for next Wednesday: From Binary Equiprobable Case till End of Chapter 3 (13p).

Problem 1. *(Non-coherent Detection)*
This problem describes a communication scheme you may found useful for your project in order to do sound communication, but for sure, you have to do some modifications to make it work.

A transmitter and receiver pair communicate over a DAWGN (Discrete Additive White Gaussian Noise) channel using OOK (on + off keying) signaling. To send bit 0, the transmitter is simply off and doesn't send any signal. Hence the receiver receives pure noise from the channel. To send bit 1, the transmitter uses a cosinusoid with frequency $\tilde{f}_0$. After sampling the received signal at suitable intervals, we can assume that the received signal can be modeled as $r_i = A\cos(2\pi f_0 i + \theta) + Z_i, \quad i = 1, 2, ..., N-1$, where A is the amplitude of the signal at the receiver, which is a fixed number and obviously $A = 0$ corresponds to pure noise case. f_0 is the discrete frequency of the signal which is between 0 and $\frac{1}{2}$, θ is a random phase introduced by the channel and $Z_i, \quad i = 0, 1, ..., N-1$ is the additive noise of the channel which we assume to be i.i.d $\mathcal{N}(0, \sigma^2)$.

1. First assume that we don't have any phase uncertainty. Derive the MAP rule and find the error probability as a function of the $SNR \triangleq \frac{A^2}{\sigma^2}$ (signal to noise ratio) and N. **Hint:** you may use the following approximations for $0 < f_0 < \frac{1}{2}$

$$\frac{1}{N}\sum_{i=0}^{N-1}\cos(2\pi f_0 i) \approx 0, \quad \frac{1}{N}\sum_{i=0}^{N-1}\sin(2\pi f_0 i) \approx 0$$
$$\frac{1}{N}\sum_{i=0}^{N-1}\cos^2(2\pi f_0 i) \approx \frac{1}{2}, \quad \frac{1}{N}\sum_{i=0}^{N}\cos(2\pi f_0 i)\sin(2\pi f_0 i) \approx 0.$$

2. Can you intuitively say what the MAP rule is trying to do to extract bit 0 and 1 from the channel noise ?

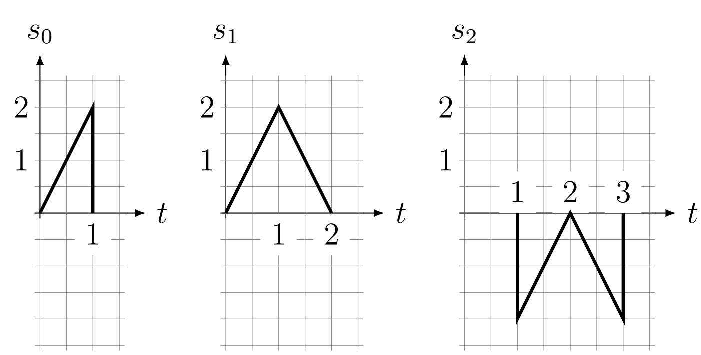
3. Now assume that you have the same MAP rule as you derived in part 1. Also assume that the transmitter sends bit 1 (cosinusoid signal) but unfortunately this time what you receive is $r_i = A\cos(2\pi f_0 i + \theta_0) + Z_i$, where θ_0 is a fixed shift. In this case derive the conditional error probability under H_1 as a function of the SNR, N, and θ_0.

4. Argue that the conditional error probability under H_0 does not depend on θ_0 and use this fact to obtain an expression for the mean error probability as a function of SNR, N, and θ_0.

5. Argue that if the receiver does not pay attention to phase uncertainty, the mean error probability can potentially be very large. This problem really happens in practice. For example in your system for sound communication, because of the propagation delay of the sound between the two laptops, which essentially cannot be controlled exactly, a random phase is introduced. In other words, even if you send $\cos(2\pi f_0 n)$, the receiver may receive $\cos(2\pi f_0 n + \theta_0)$.
For example the speed of propagation of the sound in the air is 340 m/s. If we use 1kHz signal for communication, a simple calculation shows that 1 cm uncertainty ends up in 10 degrees phase change. Hence you see that this effect can be really annoying. Think about possible ways that you can use to solve this problem. How can you derive phase-robust MAP rules ?

Problem 2. *(Gram-Schmidt Procedure On Tuples)*
Use the Gram-Schmidt orthonormalization procedure to find an orthonormal basis for the subspace spanned by the vectors $\beta_1, \dots, \beta_4$ where $\beta_1 = (1,0,1,1)^T$, $\beta_2 = (2,1,0,1)^T$, $\beta_3 = (1,0,1,-2)^T$, and $\beta_4 = (2,0,2,-1)^T$.

Problem 3. *(Gram-Schmidt Procedure on Waveforms)*
Consider the following functions $s_0(t)$, $s_1(t)$ and $s_2(t)$.



1. Using the Gram-Schmidt procedure, determine an orthonormal basis $\{\phi_0(t),\ \phi_1(t)\ ,\phi_2(t)\}$.for the space spanned by $\{s_0(t), s_1(t), s_2(t)\}$.

2. Let $\mathbf{v}_1 = (3, -1, 1)^T$ and $\mathbf{v}_2 = (-1, 2, 3)^T$ be two n-tuples of coefficients for the representation of $v_1(t)$ and $v_2(t)$ respectively. What are the signals $v_1(t)$ and $v_2(t)$? (You can simply draw a detailed graph.)

3. Compute the inner product $< v_1(t), v_2(t) >$.

4. Find the inner product $< \mathbf{v}_1, \mathbf{v}_2 >$. How does it compare to the rseult you have found in the previous question?

5. Find the norms $||v_1(t)||$ and $||\mathbf{v}_1||$ and compare them.

Problem 4. *(Matched Filter Intuition)*
In this problem, we develop some further intuition about matched filters. We have seen that an optimal receiver front end for the signal set $\{s_j(t)\}_{j=0}^{m-1}$ reduces the received (noisy) signal $R(t)$ to the m real numbers $< R, s_j >$, $j = 0, \ldots, m-1$. We gain additional intuition about the operation $< R, s_j >$ by considering

$$R(t) \;=\; s(t) + N(t), \tag{1}$$

where $N(t)$ is additive white Gaussian noise of power spectral density $N_0/2$ and $s(t)$ is an arbitrary but fixed signal. Let $h(t)$ be an arbitrary waveform, and consider the receiver operation

$$Y \;=\; < R, h > = < s, h > + < N, h > . \tag{2}$$

The signal-to-noise ratio (SNR) is thus

$$SNR \;=\; \frac{| < s, h > |^2}{E\left[| < N, h > |^2\right]}. \tag{3}$$

Notice that the SNR is not changed when $h(t)$ is multiplied by a constant. Therefore, we assume that $h(t)$ is a unit energy signal and denote it by $\phi(t)$. Then,

$$E\left[| < N, \phi > |^2\right] \;=\; \frac{N_0}{2}. \tag{4}$$

1. Use Cauchy-Schwarz inequality to give an upper bound on the SNR. What is the condition for equality in the Cauchy-Schwarz inequality? Find the $\phi(t)$ that maximizes the SNR. What is the relationship between the maximizing $\phi(t)$ and the signal $s(t)$?

2. Let $s = (s_1, s_2)^T$ and use calculus (instead of the Cauchy-Schwarz inequality) to find the $\phi = (\phi_1, \phi_2)^T$ that maximizes $< s, \phi >$ subject to the constraint that ϕ has unit energy.

3. Hence to maximize the SNR, for each value of t we have to weigh (multiply) $R(t)$ with $s(t)$ and then integrate. Verify with a picture (convolution) that the output at time T of a filter with input $s(t)$ and impulse response $h(t) = s(T-t)$ is indeed $\int_0^T s^2(t)dt$.

4. We may also look at the situation in terms of Fourier transforms. Write out the filter operation in the frequency domain.

Problem 5. *(AWGN Channel and Sufficient Statistic)*
In this problem we consider communication over the AGN channel and show that the projection of the received signal onto the signal space spanned by the signals used at the sender is not a sufficient statistic unless the noise is white.

Assume that we have only two hypotheses, i.e., $H \in \{0, 1\}$. Under $H = 0$ we send the signal $s_0(t)$ and under $H = 1$ we send $s_1(t)$. Suppose that $s_0(t)$ and $s_1(t)$ can be spanned by the orthogonal signals $\phi_0(t)$ and $\phi_1(t)$. Assume that the communication between the transmitter and the receiver is across a continuous time additive noise channel where the noise may be correlated. First assume that the additive noise is $Z(t) = N_0\phi_0(t) + N_1\phi_1(t) + N_2\phi_2(t)$ for some $\phi_2(t)$ which is orthogonal to $\phi_0(t)$ and $\phi_1(t)$ and N_0, N_1 and N_2 are independent Gaussian random variables with mean zero and variance σ^2. There is a one-to-one correspondence between the received waveform $Y(t)$ and the n-tuple of coefficients $\boldsymbol{Y} = (Y_0, Y_1, Y_2)$ where $Y_i =< \phi_i(t), Y(t) >$. $s_0(t)$ and $s_1(t)$ can be represented by $\boldsymbol{s}_0 = (\boldsymbol{s}_{00}, \boldsymbol{s}_{01}, 0)$ and $\boldsymbol{s}_1 = (s_{10}, s_{11}, 0)$ in the orthonormal basis $\{\phi_0(t), \phi_1(t), \phi_2(t)\}$.

1. Show that using vector representation, the new hypothesis testing problem can be stated as $H = i \; : \; \boldsymbol{Y} = \boldsymbol{s}_i + \boldsymbol{N} \quad i = 0, 1$ in which $\boldsymbol{N} = (N_0, N_1, N_2)$.

2. Use the Neyman-Fisher factorization theorem to show that Y_0, Y_1 is a sufficient statistic. Also show that Y_2 contains only noise under both hypotheses.

Now assume that N_0 and N_1 are independent Gaussian random variables with mean 0 and variance σ^2 but $N_2 = N_1$. In other words, noise has correlated components. We want to show that in this case Y_0, Y_1 is not a sufficient statistic. In other words Y_2 is also useful to minimize the probability of the error in the underlying hypothesis testing problem. In the following parts, for simplicity assume that $\boldsymbol{s}_0 = (1, 0, 0)$ and $\boldsymbol{s}_1 = (0, 1, 0)$ and $H = 0$ and $H = 1$ are equiprobable.

1. Find the the minimum probability of the error when we have access only to Y_0, Y_1.

2. Show that by using Y_0, Y_1 and Y_2 you can reduce the probability of the error. Specify the decision rule that you suggest and evaluate its probability of error. Is your receiver optimum in the sense of minimizing the probability of the error?

3. Argue that in this case (Y_0, Y_1) is not a sufficient statistic.