Solution de la série 5
Traitement quantique de l'information II

Exercice 1

1a)

$$1 \otimes CU \, |x, y, z\rangle = |x\rangle \otimes |y\rangle \otimes U^y \, |z\rangle$$

$$(\text{CNOT} \otimes 1)(1 \otimes CU) \, |x, y, z\rangle = |x\rangle \otimes |y \oplus x\rangle \otimes U^y \, |z\rangle$$

$$(1 \otimes CU^+)(\text{CNOT} \otimes 1)(1 \otimes CU) \, |x, y, z\rangle = |x\rangle \otimes |y \oplus x\rangle \otimes (U^+)^{x \oplus y} U^y \, |z\rangle$$

$$(\text{CNOT} \otimes 1)(1 \otimes CU^+)(\text{CNOT} \otimes 1)(1 \otimes CU) \, |x, y, z\rangle = |x\rangle \otimes |y\rangle \otimes (U^+)^{x \oplus y} U^y \, |z\rangle$$

Dernière $CU(\text{CNOT}\otimes 1)(1\otimes CU^+)(\text{CNOT}\otimes 1)(1\otimes CU) \, |x, y, z\rangle = |x\rangle \otimes |y\rangle \otimes U^x (U^+)^{x \oplus y} U^y \, |z\rangle$

Notons que $U^{2xy} = U^x U^{+^{x\oplus y}} U^y$. En effet
si $x = 1, y = 1$ on a $U^2 = U(U^+)^0 U$
si $x = 1, y = 0$ on a $U^0 = UU^+U^0$
si $x = 0, y = 1$ on a $U^0 = U^0U^+U$
si $x = 0, y = 0$ on a $U^0 = U^0(U^+)^0U^0$ cqfd.

1b) Pour réaliser un CCNOT il faut prendre $U^2 = \text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Donc $U = \sqrt{\text{NOT}}$. On peut verifier que

$$\begin{pmatrix} 1 & +i \\ +i & 1 \end{pmatrix}^2 = 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Donc,

$$\sqrt{\text{NOT}} = \frac{1}{\sqrt{2i}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Exercice 2

2a)

$$H|0\rangle \otimes |u\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |u\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |u\rangle$$

$$CUH|0\rangle \otimes |u\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |u\rangle + \frac{1}{\sqrt{2}}e^{2\pi i\varphi}|1\rangle \otimes |u\rangle$$

$$\begin{aligned} HCUH|0\rangle \otimes |u\rangle &= \frac{1}{2}(|0\rangle + |1\rangle) \otimes |u\rangle + \frac{e^{2\pi i\varphi}}{2}(|0\rangle - |1\rangle) \otimes |u\rangle \\ &= \frac{1+e^{2\pi i\varphi}}{2}|0\rangle \otimes |u\rangle + \frac{1-e^{2\pi i\varphi}}{2}|1\rangle \otimes |u\rangle \\ &= e^{\pi i\varphi}(\cos\pi\varphi\,|0\rangle \otimes |u\rangle - i\sin\pi\varphi\,|1\rangle \otimes |u\rangle) \end{aligned}$$

2b)

$$\text{Prob}(0) = \cos^2\pi\varphi \quad \text{et} \quad \text{Prob}(1) = \sin^2\pi\varphi$$

2c) Si on applique U^k au lieu de U on trouve la sortie :

$$e^{i\pi k\varphi}(\cos(\pi k\varphi)\,|0\rangle \otimes |u\rangle - i\sin(\pi k\varphi)\,|1\rangle \otimes |u\rangle)$$

Si $\varphi = \frac{\varphi_1}{2} + \frac{\varphi_2}{2^2} + ... + \frac{\varphi_{t-1}}{2^{t-1}} + \frac{\varphi_t}{2^t}$ en prenant $k = 2^{t-1}$ on observe 0 avec probabilité

$$\text{Prob}(0) = \cos^2(\pi\varphi_{t-1} + \frac{\pi\varphi_t}{2}) = \cos^2(\frac{\pi\varphi_t}{2}) = \begin{cases} 1 & \text{si } \varphi_t = 0 \\ 0 & \text{si } \varphi_t = 1 \end{cases}$$