ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 24 Signal Processing for Communications
Homework 9 May 9, 2011, INF 1 - 10:15-12:00

Problem 1 (DFT Revisit).

(i) Define $Y[k] = X[k](-1)^k = X[k]e^{j2\pi\frac{k}{N}(N/2)}$, then $y[n] = x[(n - \frac{N}{2}) \bmod N]$.

On the other hand, $X_1[k] = \text{real}\{Y[k]\} = \frac{1}{2}(Y[k] + Y^*[k])$. Since $x[n]$ is real and therefore $y[n]$ is real, $Z[k] = Y^*[k] = Y[-k \bmod N]$ and then, $z[n] = y[-n \bmod N]$.

Thus, $x_1[n] = y[n] + y[-n \bmod N] = x[(n - \frac{N}{2}) \bmod N] + x[(-n - \frac{N}{2}) \bmod N]$.

(ii) $x_1[n] = \frac{1}{2}(y[n] + y[-n \bmod N])$ and $x_2[n] = \frac{1}{2}(y[n] - y[-n \bmod N])$ and since the DFT is linear function, we can say that

$$X_1[k] = \frac{1}{2}(Y[k] + Y[-k \bmod N]),$$

and

$$X_2[k] = \frac{1}{2}(Y[k] - Y[-k \bmod N]).$$

Problem 2 (Limits of Z-transform).

(i) $X(1) = \sum_{n=-\infty}^{\infty} x[n] =^{(a)} \sum_{n=0}^{\infty} x[n]$
(a) is correct since $x[n]$ is causal. It shows the limit of the series $\sum_{n=0}^{\infty} x[n]$. If the ROC of $X(z)$ contains the unit circle, then it has limit and the limit is equal to $X(1)$.

(ii) $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = x[0]$.

(iii) $\lim_{z\to\infty} z(X(z) - x[0]) = x[1]$. The result follows from the fact that :

$$\begin{aligned}
\lim_{z\to\infty} z(X(z) - x[0]) &= \lim_{z\to\infty} z\left(\sum_{n=0}^{\infty} x[n]z^{-n} - x[0]\right) \\
&= \lim_{z\to\infty} z\left(\sum_{n=1}^{\infty} x[n]z^{-n}\right) \\
&= \lim_{z\to\infty} \sum_{n=1}^{\infty} x[n]z^{-(n-1)} \\
&= \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n+1]z^{-n} = x[1].
\end{aligned}$$

(iv) $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \Rightarrow \frac{dX(z)}{dz} = \sum_{n=0}^{\infty} -nx[n]z^{-(n+1)}$.
Therefore, $-z\frac{dX(z)}{dz} = \sum_{n=0}^{\infty} nx[n]z^{-n}$.
It is the z-tranform of $nx[n]$.

(v) from (iv), $-z\frac{dX(z)}{dz} = \sum_{n=0}^{\infty} nx[n]z^{-n} = \sum_{n=1}^{\infty} nx[n]z^{-n}$.
Hence,

$$\lim_{z\to\infty} -z^2\frac{dX(z)}{dz} = \lim_{z\to\infty}\sum_{n=1}^{\infty} nx[n]z^{-(n-1)} = \lim_{z\to\infty}\sum_{n=0}^{\infty}(n+1)x[n+1]z^{-n} = x[1]$$

Problem 3 (Stochastic Processes)**.**

(i)

$$\begin{aligned} m_x &= E[x[n]] = E[\sin(\omega n+\theta)] = E[\sin(\omega n)\cos(\theta)+\cos(\omega n)\sin(\theta)] \\ &= \sin(\omega n)E[\cos(\theta)] + \cos(\omega n)E[\sin(\theta)]. \end{aligned}$$

$$E[\sin(\theta)] = \int_{-\infty}^{\infty}\sin(\theta)f_\theta(\theta)d\theta = \int_0^{2\pi}\sin(\theta)\frac{1}{2\pi}d\theta = \frac{-1}{2\pi}\cos(\theta)\Big|_0^{2\pi} = 0.$$

In the same manner, $E[\cos(\theta)] = 0$.

$$R_X[\ell,k] = E[X[\ell]X[k]] = E\{\sin(\omega\ell+\theta)\sin(\omega k+\theta)\}.$$

We know that $\sin(\varphi_1)\sin(\varphi_2) = \frac{1}{2}(\cos(\varphi_1-\varphi_2)-\cos(\varphi_1+\varphi_2))$. Thus,

$$\begin{aligned} R_X[\ell,k] &= E\left\{\frac{1}{2}[\cos(\omega(\ell-k)) - \cos(\omega(\ell+k)+2\theta)]\right\} \\ &= \frac{1}{2}E[\cos(\omega(\ell-k))] - \frac{1}{2}E\{\cos(\omega(\ell+k)+2\theta)\} \\ &= \frac{1}{2}\cos(\omega(\ell-k)) - \frac{1}{2}E\{\cos(\omega(\ell+k)+2\theta)\} = \frac{1}{2}\cos(\omega(\ell-k)). \end{aligned}$$

The last equality is due to

$$\begin{aligned} E\{\cos(\omega(\ell+k)+2\theta)\} &= E\{\cos(\omega(\ell+k))\cos(2\theta) - \sin(\omega(\ell+k))\sin(2\theta)\} \\ &= \cos(\omega(\ell+k))E\{\cos(2\theta)\} - \sin(\omega(\ell+k))E\{\sin(2\theta)\}. \end{aligned}$$

and $E\{\cos(2\theta)\} = \int_0^{2\pi}\cos(2\theta)\frac{1}{2\pi}d\theta = \frac{1}{4\pi}\sin(2\theta)\Big|_0^{2\pi} = 0$.
Similarly, $E\{\sin(2\theta)\} = 0$.

Since m_X is fixed and $R_X[\ell,k]$ is only a function of $\ell - k$, we can say that $x[n]$ is a wide-sense stationary signal.

(ii) Let's first compute the impulse response of this filter.

$$h[n] = \delta[n] + \beta\delta[n-1]$$

Therefore,

$$H(e^{j2\pi f}) = 1 + \beta e^{-j2\pi f}.$$

On the other hand,

$$P_X(e^{j2\pi f}) = FT\{R_X[k]\} = \frac{1}{2j}\left[\tilde{\delta}(2\pi f-\omega) - \tilde{\delta}(2\pi f+\omega)\right].$$

Therefore,

$$\begin{aligned} P_Y(e^{j2\pi f}) &= |H(e^{j2\pi f})|^2 P_X(e^{j2\pi f}) \\ &= |H(e^{j2\pi f})|^2 \frac{1}{2j}\left[\tilde{\delta}(2\pi f-\omega)-\tilde{\delta}(2\pi f+\omega)\right] \\ &= |H(e^{j\omega})|^2 \frac{1}{2j}\left[\tilde{\delta}(2\pi f-\omega)-\tilde{\delta}(2\pi f+\omega)\right]. \end{aligned}$$

(iii) We should compute $P_X(e^{j2\pi f})$:

$$\begin{aligned} P_X(e^{j2\pi f}) &= \sum_{k=-\infty}^{\infty} R_X[k]e^{-j2\pi fk} = \sigma^2 \sum_{k=-\infty}^{-1} \alpha^{-k} e^{-j2\pi fk} + \sigma^2 \sum_{k=0}^{\infty} \alpha^k e^{-j2\pi fk} \\ &= \sigma^2\left(\frac{\alpha e^{j2\pi f}}{1-\alpha e^{j2\pi f}} + \frac{1}{1-\alpha e^{-j2\pi f}}\right) = \sigma^2\left(\frac{1-\alpha^2}{1+\alpha^2-\alpha(e^{-j2\pi f}+e^{j2\pi f})}\right) \end{aligned}$$

More over,

$$\begin{aligned} |H(e^{j2\pi f})|^2 = |1+\beta e^{-j2\pi f}|^2 &= |1+\beta\cos(2\pi f) - j\beta\sin(2\pi f)|^2 \\ &= (1+\beta^2+2\beta\cos(2\pi f)). \end{aligned}$$

Thus,

$$P_Y(e^{j2\pi f}) = (1+\beta^2+2\beta\cos(2\pi f))\sigma^2\left(\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(2\pi f)}\right).$$

(iv) $Y[n]$ corresponds to a white noise, if it power spectral density is a constant value for all frequencies. Therefore,

$$P_Y(e^{j2\pi f}) = (1+\beta^2+2\beta\cos(2\pi f))\sigma^2\left(\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(2\pi f)}\right) = \text{const.}$$

if and only if

$$\frac{1+\beta^2+2\beta\cos(2\pi f)}{1+\alpha^2-2\alpha\cos(2\pi f)} = \text{const.}$$

Hence, we can conclude that $\beta = -\alpha$.

Problem 4 (Min. Mean Squared Error Estimator*)**.**

(i) We should verify the following three properties of inner product :

- Positivity :

$$\langle u,u\rangle = \int uu^* P_{X,Y}(x,y)dxdy = \int |u|^2 P_{X,Y}(x,y)dxdy \geq 0$$

- Linearity :

$$\begin{aligned} \langle au+bw,v\rangle = E((au+bw)v^*) &= aE(uv^*)+bE(wv*) \\ &= a\langle u,v\rangle + b\langle w,v\rangle \end{aligned}$$

The above equalities are due to linearity of expectation function.

- Conjugate symmetry :

$$\langle u, v\rangle = E(uv^*) = (E(u^*v))^* = (\langle v, u\rangle)^*$$

(ii) Since it is unbiased estimator,

$$E(X) = E(\hat{X}) = E(aY + b) = aE(Y) + b \tag{1}$$

Since it is minimum mean squared estimator,

$$\begin{aligned} E\left\{(X - \hat{X})^2\right\} &= E\left\{X^2 + \hat{X}^2 - 2X\hat{X}\right\} \\ &= E(X^2) + E\left\{\hat{X}^2 - 2X\hat{X}\right\} \end{aligned}$$

$E(X^2)$ is fixed and we should minimize the second component :

$$\begin{aligned} E\left\{\hat{X}^2 - 2X\hat{X}\right\} &= E\left\{(aY + b)^2 - 2X(aY + b)\right\} \\ &= E((a^2Y^2 + b^2 + 2abY) - 2aXY - 2bX) \\ &= a^2E(Y^2) + \underbrace{b^2 + 2abE(Y) - 2bE(X)}_{-b^2 \text{ from (1)}} - 2aE(XY) \\ &= a^2E(Y^2) - 2aE(XY) - b^2 \end{aligned}$$

We know that $b = E(X) - aE(Y) = m_X - am_Y$. Thus,

$$E\left\{\hat{X}^2 - 2X\hat{X}\right\} = a^2E(Y^2) - 2aE(XY) - (m_X - am_Y)^2 = f(a)$$

To minimize $E\left\{\hat{X}^2 - 2X\hat{X}\right\} = f(a)$, we can take the derivative from $f(a)$ and set it equal to zero,

$$\begin{aligned} f'(a) &= 2aE(Y^2) - 2E(XY) + 2m_Y(m_X - am_Y) = 0 \\ \Rightarrow a &= \frac{E(XY) - m_Xm_Y}{E(Y^2) - m_Y^2}, \qquad b = m_X - am_Y \end{aligned}$$

(iii) Shortly, the subspace of random variable Y contains Y and all continuous functions $f(y)$. Assume that $p(y)$ is the minimum mean squared estimator, i.e.

$$\arg\min_{f(y)} \langle x - f(y), x - f(y)\rangle = p(y)$$

According to projection theorem, since $x - p(y)$ has the minimum norm for all members of subspace. $p(y)$ is projection of x on that subspace and, as we know, it is the projection iff

$$\langle x - p(y), f(y)\rangle = 0$$

($x - p(y)$ is orthogonal with all the members of subspace)

According to hint 2, $E(X|Y) = g(Y)$ has such property and, therefore, $g(Y) = E(X|Y)$ is the projection of X on the subspace of Y and it is the best minimum squared error estimator.