ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 12 Introduction to Communication Systems
Solution 5 October 26, 2010, SG1 – 15:15-17:00

Problem 1. Write

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \\ &\leq (n+1) \max_k \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

Let $a_k = \binom{n}{k} p^k (1-p)^{n-k}$ and we have $\frac{a_k}{a_{k-1}} = \frac{p(n-k+1)}{k(1-p)}$ so the maximum of a_k is when $k = np$. By letting $p = \frac{k}{n}$, we have

$$\begin{aligned} 1 &\leq (n+1) \binom{n}{np} (np)^k (1-p)^{n(1-p)} \\ &= (n+1) \binom{n}{k} (k)^k (1-\frac{k}{n})^{n(1-\frac{k}{n})} \\ &= (n+1) \binom{n}{k} 2^{-nh_2(\frac{k}{n})}. \end{aligned}$$

Problem 2. (a) $M = 1 + \alpha(K-1)$.

(b) It's straight forward.

(c) The expression $E[\text{length}(W)]$ changes to $E[\text{length}(W')] = E[\text{length}(W)] + p$. The expression $H(W)$ changes to $H(W) + p \log p - \sum_x px \log(px) = H(W) + -pH(X)$. Finally,

$$\begin{aligned} H(W') &= H(W) + pH(X) \\ &= H(X)E[\text{length}(W)] + H(X)p \\ &= H(X)[E[\text{length}(W) + p] \\ &= H(X)E[\text{length}(W')]. \end{aligned}$$

Problem 3. The decoding is “lalalal“.

Problem 4. (a) $\log(n+1)$ bits are needed to be reserved for the description of k.

$$N = \binom{n}{k}$$

(b) We need $\lceil \log \binom{n}{k} \rceil$ bits to describe which of the N sequences we are decoding with.

(c)

$$\begin{aligned} l(x^n) &= \lceil \log(n+1) \rceil + \lceil \log \binom{n}{k} \rceil \\ &\leq \log(n) + \log \binom{n}{k} + 3 \end{aligned} \quad (1)$$

Since $\binom{n}{k} \leq 2^{nH(\frac{k}{n})}\sqrt{\frac{n}{\pi k(n-k)}}$

$$\begin{aligned} \log \binom{n}{k} &\leq \log \left(2^{nH(\frac{k}{n})}\sqrt{\frac{n}{\pi k(n-k)}}\right) \\ &= nH(\frac{k}{n}) + \log \sqrt{\frac{n}{\pi k(n-k)}} \end{aligned} \tag{2}$$

So

$$l(x^n) \leq \log n + nH(\frac{k}{n}) - \frac{1}{2}\log n + \log \sqrt{\frac{n^2}{\pi k(n-k)}} + 3 \tag{3}$$

$$= \frac{1}{2}\log n + nH(\frac{k}{n}) \underbrace{-\frac{1}{2}\log \left(\pi \frac{k}{n} \times \frac{n-k}{n}\right) + 3}_{\text{does not grow with } n} \tag{4}$$

(d)

$$\frac{l(x^n) - l^*(x^n)}{l^*(x^n)} \leq \frac{\frac{1}{2}\log n + nH(\frac{k}{n}) + O(c) - nH(\frac{k}{n})}{nH(\frac{k}{n})} = \frac{1/2 \log n}{nH(\frac{k}{n})} + \frac{O(c)}{nH(\frac{k}{n})} \tag{5}$$

$$\lim_{n\to\infty} \frac{l(x^n) - l^*(x^n)}{l^*(x^n)} = 0 \tag{6}$$