EPFL Winter Semester 2007/2008
Handout # 26, Nov 14, 2007

Solutions to the Midterm Exam

Problem 1

(a)

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{0} 2^n e^{-j\omega n} + j\sum_{n=0}^{\infty} 3^{-n}e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} 2^{-n}e^{j\omega n} + j\sum_{n=0}^{\infty} 3^{-n}e^{-j\omega n} \\ &= \frac{1}{1-\frac{1}{2}e^{j\omega}} + \frac{j}{1-\frac{1}{3}e^{-j\omega}}. \end{aligned}$$

(b) Using the DTFT properties, we find that

$$\begin{aligned} x^*[n] &\overset{\text{DTFT}}{\longleftrightarrow} X^*(e^{-j\omega}) \\ x^*[-n] &\overset{\text{DTFT}}{\longleftrightarrow} X^*(e^{j\omega}). \end{aligned}$$

Applying this to (a), we find that

$$\begin{aligned} Y(e^{j\omega}) &= X^*(e^{j\omega}) + X(e^{j\omega}) \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{j}{1-\frac{1}{3}e^{j\omega}} + \frac{1}{1-\frac{1}{2}e^{j\omega}} + \frac{j}{1-\frac{1}{3}e^{-j\omega}}, \end{aligned}$$

where we have used the fact that $(ab)^* = a^*b^*$, $(a/b)^* = a^*/b^*$, etc.

(c) Noting that $y[n-1] = x^*[-(n-1)] + x[n-1] = z[n]$, we find that $Z(e^{j\omega}) = e^{-j\omega}Y(e^{j\omega})$, where $Y(e^{j\omega})$ is the result found in (b).

Problem 2

(a) Note that the whole system is just a cascade combination of two systems, one with input $x[n]$ and output $w[n]$, and the other one with input $w[n]$ and output $y[n]$. For the first system, we can write

$$w[n] = h_1[n] * (x[n] + h_2[n] * w[n]) = \delta[n-1] * (x[n] + 9\delta[n-1] * w[n]).$$

By taking the z-transform of the both sides, we have

$$W(z) = z^{-1} \cdot (X(z) + 9z^{-1}W(z)).$$

By isolating $W(z)$, we have

$$W(z) = \frac{z^{-1}}{1 - 9z^{-2}} X(z).$$

For the second part of the system, we can simply write

$$y[n] = h_3[n] * w[n] + h_4[n] * w[n] = 2\delta[n-2] * w[n] + 3\delta[n-1] * w[n]$$

or

$$Y(z) = 2z^{-2}W(z) + 3z^{-1}W(z) = (2z^{-2} + 3z^{-1})W(z) = (2z^{-2} + 3z^{-1})\frac{z^{-1}}{1 - 9z^{-2}} X(z).$$

Therefore, the whole system transfer function, $H(z)$, is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-3} + 3z^{-2}}{1 - 9z^{-2}}$$

Note that the poles (roots of the denominator) of the system are $z_1 = +3$, and $z_2 = -3$, so there are two possible regions of cenvergence for this system, which are $ROC_1 : |z| < 3$, and $ROC_2 : |z| > 3$. However, it is clear from the structure of the system that the output only depends on the past of the input. This implies the system is causal. So its ROC should be outside of the ring (and include the infinity), *i.e.*, $ROC : |z| > 3$.

(b) Form part (a), we have

$$\frac{Y(z)}{X(z)} = \frac{2z^{-3} + 3z^{-2}}{1 - 9z^{-2}}$$

or

$$Y(z) - 9z^{-2}Y(z) = 2z^{-3}X(z) + 3z^{-2}X(z).$$

By taking the inverse z-transform, we have

$$y[n] - 9y[n-2] = 2x[n-3] + 3x[n-2],$$

which gives us the difference equation as

$$y[n] = 9y[n-2] + 2x[n-3] + 3x[n-2].$$

(c) Note that $h[n]$ is the inverse z-transform of the $H(z)$. In order to compute that, we need to find the fractional expansion of $H(z)$. By dividing the numerator by the denominator, we have

$$\begin{aligned} H(z) &= \frac{2z^{-3} + 3z^{-2}}{1 - 9z^{-2}} = \frac{(-\frac{2}{9}z^{-1} - \frac{1}{3})(1 - 9z^{-2}) + (\frac{2}{9}z^{-1} + \frac{1}{3})}{1 - 9z^{-2}} \\ &= -\frac{2}{9}z^{-1} - \frac{1}{3} + \frac{1}{9}\left(\frac{2z^{-1} + 3}{1 - 9z^{-2}}\right). \end{aligned}$$

The roots of the denominator of $H(z)$ are

$$1 - 9z^{-2} = 0 \qquad \Rightarrow \qquad z^{-1} = \pm\frac{1}{3} \qquad \Rightarrow \qquad z = \pm 3.$$

So, $H(z)$ can be written as

$$H(z) = -\frac{2}{9}z^{-1} - \frac{1}{3} + \frac{1}{9}\left(\frac{A}{1-3z^{-1}} + \frac{B}{1+3z^{-1}}\right)$$

where A and B should satisfy

$$\begin{aligned} A + B &= 3 \\ 3(A - B) &= 2. \end{aligned}$$

Thus $A = \frac{11}{6}$ and $B = \frac{7}{6}$. Finally we have

$$H(z) = -\frac{2}{9}z^{-1} - \frac{1}{3} + \frac{11}{54}\frac{1}{1-3z^{-1}} + \frac{7}{54}\frac{1}{1+3z^{-1}},$$

where we can easily compute the inverse z-transform as

$$h[n] = -\frac{2}{9}\delta[n-1] - \frac{1}{3}\delta[n] + \frac{11}{54}3^n u[n] + \frac{7}{54}(-3)^n u[n].$$

(d) We already discussed in (a) that the system is causal.

(e) As we saw in (a), the region of convergence of the system is $ROC : |z| > 3$, which does not contain the unit circle $|z| = 1$ (*i.e.*, the z-trasform is not defined for $|z| = 1$, or the DTFT of the system is not defined). Thus the overall system is not BIBO stable. This can be also seen through the impulse response of the system $h[n]$, where if we choose a bounded input $x[n] = \delta[n]$, the ouput will exponentially increase with n, and is not bounded.

Problem 3

(a) The quick method: Noting that the left-hand side of the equation is the DFS coefficient of $\tilde{x}^*[n]\tilde{y}[n]$ evaluated at $k = 0$, we can use the modulation property $\tilde{x}^*[n]\tilde{y}[n] \stackrel{\text{DFS}}{\longleftrightarrow} \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}^*[-(k-l)]\tilde{Y}[l]$ with $k = 0$, thereby obtaining the desired result.

Alternatively, we have

$$\begin{aligned} \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}^*[k]\tilde{Y}[k] &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}^*[k]\sum_{n=0}^{N-1}\tilde{y}[n]e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1}\tilde{y}[n]\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}^*[k]e^{-j\frac{2\pi}{N}kn}\right) \\ &= \sum_{n=0}^{N-1}\tilde{y}[n]\left(\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}^*[-l]e^{j\frac{2\pi}{N}ln}\right) \\ &= \sum_{n=0}^{N-1}\tilde{x}^*[n]\tilde{y}[n]. \end{aligned}$$

(b) The period of both $\tilde{x}[n]$ and $\tilde{y}[n]$ is 8. The DFS of $\tilde{x}[n]$ and $\tilde{y}[n]$ are, respectively,

$$\tilde{X}[k] = 8\tilde{\delta}[k] + \frac{8}{2}(\tilde{\delta}[k-1] + \tilde{\delta}[k-7]),$$
$$\tilde{Y}[k] = \frac{8}{2j}(e^{j\pi/4}\tilde{\delta}[k-1] - e^{-j\pi/4}\tilde{\delta}[k+1].$$

Using the result of (a), we have

$$\begin{aligned}\frac{1}{8}\sum_{k=0}^{N-1}\tilde{X}^*[k]\tilde{Y}[k] &= \frac{4}{2j}e^{j\pi/4} - \frac{4}{2j}e^{-j\pi/4}\\ &= 4\sin(\pi/4) = 4/\sqrt{2} = 2\sqrt{2}.\end{aligned}$$

Problem 4

(a) Suppose the claim holds, i.e., there is $x[n] \neq y[n]$ such that $X_1[k] = Y_1[k]$ for all $k = 0, \ldots, M-1$. Then we could write

$$\begin{aligned}0 &= X_1[k] - Y_1[k]\\ &= \sum_{n=0}^{M-1}(x[n] - y[n])e^{-j\frac{2\pi}{M}kn}.\end{aligned}$$

This would mean that the M-point DFT of the nonzero sequence $x[n] - y[n]$, which has finite support of length M, is identically zero. This, however, is impossible because the DFT is a reversible transform, and so a nonzero finite length sequence cannot have the same DFT as the all-zero sequence. The claim must be false.

(b) If $N < M$, the claim can be true. For example, suppose $M = 2N$, take an arbitrary (nonzero) $x[n]$, and let $y[n]$ be defined as

$$y[n] = \begin{cases} (1/2)x[n], & 0 \le n \le N-1, \\ (1/2)x[n-N], & N \le n \le M-1. \end{cases}$$

Then, clearly, $x[n] \neq y[n]$. On the other hand, we have

$$\begin{aligned}X_2[k] - Y_2[k] &= \sum_{n=0}^{2N-1} x[n]y[n]e^{-j\frac{2\pi}{N}kn}\\ &= \frac{1}{2}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} - \frac{1}{2}\sum_{n=N}^{2N-1} x[n-N]e^{-j\frac{2\pi}{N}kn}\\ &= \frac{1}{2}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} - \frac{1}{2}\sum_{n'=0}^{N-1} x[n']e^{-j\frac{2\pi}{N}k(n'+N)}\\ &= 0,\end{aligned}$$

showing that the claim can be true.

(c) Note that $z[n] = x[n-N]$. Then,

$$\begin{aligned}z[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}k(n-N)}\\ &= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j\frac{2\pi}{N}k})e^{j\frac{2\pi}{N}k(n-N)},\end{aligned}$$

where $X[k]$ is the DFT of $x[n]$ and where we have used the fact that the N-point DFT coefficients are the samples of the DTFT at $\omega = 2\pi k/N$, $k = 1, \ldots, N-1$. Now, using the property that $Z(e^{j\omega}) = e^{-j\omega N} X(e^{j\omega})$, we get

$$\begin{aligned} z[n] &= \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kN} Z(e^{j\frac{2\pi}{N}k}) e^{j\frac{2\pi}{N}k(n-N)} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} Z(e^{j\frac{2\pi}{N}k}) e^{j\frac{2\pi}{N}kn}. \end{aligned}$$

Therefore we can reconstruct $z[n]$ if we have N samples of its DTFT at $\omega = 2\pi k/N$, $k = 1, \ldots, N-1$. On the other hand, we need *at least* N samples, since $z[n]$ has in general N nonzero values. We find thus $Q = N$.

Problem 5

To see whether there is a causal system with system function $H(z)$, we need to decompose $H(z)$ into a sum of simple terms. Since the numerator is a polinomial (in z^{-1}) of a larger degree than the denominator, we start by doing a long division.

$$\left(8z^{-3} + 20z^{-2} - 10z^{-1} + 3\right) : \left(8z^{-2} - 4z^{-1} + 1\right) = z^{-1} + 3$$

with a remainder of z^{-1}. Therefore, we have

$$H(z) = 3 + z^{-1} + \frac{z^{-1}}{8z^{-2} - 4z^{-1} + 1}.$$

The first two terms are now easy to deal with. The first term has no poles and no zeros, and for all values of z, the corresponding term in $h[n]$ is $3\delta[n]$. The second term has a single pole $p_1 = 0$. Hence, its ROC is $\mathbb{C} \setminus \{0\}$. The third term needs to be further decomposed. We first find its poles, which are the zeros of $8z^{-2} - 4z^{-1} + 1$. It is clear that for any z (apart from 0 and ∞), the function $8z^{-2} - 4z^{-1} + 1$ is zero if and only if the function $8 - 4z + z^2$ is zero. Solving the quadratic function

$$z^2 - 4z + 8 = 0$$

yields $p_{2,3} = 2 \pm 2j$. Now, we do the partial fraction expansion to find the elementary terms corresponding to these two poles:

$$\begin{aligned} \frac{z^{-1}}{\left(1 - (2+2j)z^{-1}\right)\left(1 - (2-2j)z^{-1}\right)} &= \frac{A}{1 - (2+2j)z^{-1}} + \frac{B}{1 - (2-2j)z^{-1}} \\ z^{-1} &= A - A(2-2j)z^{-1} + B - B(2+2j)z^{-1}. \end{aligned}$$

Comparing the terms in z^{-1} and the constant terms separately, we obtain the system of equations

$$\begin{aligned} A + B &= 0 \\ -2(A+B) + 2j(A-B) &= 1, \end{aligned}$$

whose solution is $A = \frac{-j}{4}$ and $B = \frac{j}{4}$. Hence, the four simple terms of $H(z)$ are

$$H(z) = 3 + z^{-1} + \frac{\frac{-j}{4}}{1 - (2+2j)z^{-1}} + \frac{\frac{j}{4}}{1 - (2-2j)z^{-1}}. \quad (1)$$

(a) As mentioned before, the regions of convergence of the first two terms in (1) are $\text{ROC}_1 = \mathbb{C}$ and $\text{ROC}_2 = \mathbb{C} \setminus \{0\}$, respectively, with no choice whatsoever. For the last two terms in (1), we have two choices: $\text{ROC}_{34} = \{z : |z| > |2+2j|\}$ or $\text{ROC}'_{34} = \{z : |z| < |2+2j|\}$. If we pick the first one, we see that $\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \cap \text{ROC}_{34}$ is the outside of the circle with radius $|2+2j| = \sqrt{8}$. Therefore, there is a causal system with system function $H(z)$. The corresponding impulse response is

$$h[n] = 3\delta[n] + \delta[n-1] - \frac{j}{4}(2+2j)^n u[n] + \frac{j}{4}(2-2j)^n u[n].$$

Note: Although $h[n]$ contains complex terms, it is still real, because the two complex terms are conjugates of each other. The system is not stable, because the ROC does not contain the unit circle.

(b) Here, we try the second choice for the region of convergence of the last two terms in (1), which is $\text{ROC}'_{34} = \{z : |z| < |2+2j|\}$. Unfortunately, the overall region of convergion $\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \cap \text{ROC}'_{34}$ is not the inside of a circle (because it does not include the point 0). Hence, there is no anticausal system with system function $H(z)$.

(c) We see that the impulse response of the composition is $p[n] = 8\delta[n-3] + 20\delta[n-2] - 10\delta[n-1] + 3\delta[n]$. This system is causal, because clearly, $p[n] = 0$ for all $n < 0$.

(d) The system function $G(z)$ that we are looking for should satisfy $P(z) = H(z)G(z)$, and we can easily find that $P(z) = 8z^{-3} + 20z^{-2} - 10z^{-1} + 3$. Hence,

$$G(z) = \frac{P(z)}{H(z)} = \frac{8z^{-3} + 20z^{-2} - 10z^{-1} + 3}{\frac{8z^{-3}+20z^{-2}-10z^{-1}+3}{8z^{-2}-4z^{-1}+1}} = 8z^{-2} - 4z^{-1} + 1.$$

(e) We see that the only pole of $G(z)$ is $p_1 = 0$ (multiplicity 3). Hence, the ROC is $\mathbb{C} \setminus \{0\}$. The corresponding impulse response is hence causal and given by

$$g[n] = 8\delta[n-2] - 4\delta[n-1] + \delta[n].$$

The system is stable, because its ROC contains the unit circle.