ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 19 Signal Processing for Communications
Midterm April 19, 2011, INF 119 (A-M) INF 213 (N-Z) - 8:15-10:00

You have 2 hours. It is not necessarily expected that you finish all problems. Do not lose too much time on each problem but try to collect as many points as possible.

Closed-book, no calculators, cell-phones. Write only what is relevant to the question! Only write on the handed out pages. No exceptions. Don't forget to add your name on the top of each problem.

Good Luck!!

Name: ______________________________

Prob I	/ 50
Prob II	/ 25
Prob III	/ 30
Prob IV	/ 30
Total	/ 135

Problem 1 (A Little Bit of Everything). **Name:** ______________________________

(i) [5 pts] Let $h_1[n]$ be the real impulse response of a causal LTI system with arbitrary phase characteristics. Assume we cascade the filter $h_1[n]$ with $h_2[n] = h_1[-n]$, call the result $h[n]$. What can you tell about the phase characteristics of $h[n]$? Is this system still causal?

(ii) [10 pts] Prove or disprove: A discrete-time stable signal has always finite energy.

(iii) [5 pts] Let $f(x) = e^{-3x}$. What is $A = \int_{-\infty}^{\infty} f(x)\delta(3 - x/2)dx$?

(iv) [5 pts] Let $x[n] = u[n]$, the step function of height 1. Draw $y[n] = x[-n+3] - x[2n+1]$.

(v) [5 pts] Let H be a subset of the Hilbert space over $\mathbb{R}$ with L_2-norm, spanned by the set of functions $P = \{p_0(x), \cdots, p_{N-1}(x)\}$, where

$$p_i(x) = \begin{cases} 1 & i \leq x \leq i+1 \\ 0 & \text{otherwise} \end{cases}$$

Show that P is an orthonormal basis for the subset H.

(vi) [10 pts] Let

$$u_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Let $W = \text{span}\{u_1, u_2\}$, i.e., W is the subspace spanned by u_1 and u_2. Write y as the sum of a vector in W, and a vector orthogonal to W.

(vii) [10 pts] Consider a discrete stationary process $X[n]$ with mean zero and power spectral density $P_x(e^{j2\pi f})$. Assume that $x[n]$ is passed through an LTI filter with the frequency response $H(e^{j2\pi f})$. How shall we chose $H(e^{j2\pi f})$ so that the output process $Y[n]$ has the property

$$\mathbb{E}[Y[n]Y[m]] = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

Hint: Think in the frequency domain. Find a relationship between $H(e^{j2\pi f})$ and $P_X(e^{j2\pi f})$.

Solution:

(i) If $h_1[n]$ has DTFT $H_1(e^{-2\pi jf})$ and is real, then $h_2[n] = h_1[-n]$ has associated DTFT $H^*(e^{-2\pi jf})$. Hence, the concatenation of these two systems has DTFT $|H_1(e^{-2\pi jf})|^2$. This has phase equal to 0 and is not causal in general.

(ii) Stability means that $\sum_n |h[n]| < \infty$ (i.e., $h[n]$ is in l_1), whereas finite energy means that $\sum_n |h[n]|^2 < \infty$ (i.e., $h[n]$ is in l_2. Indeed, $h[n] \in l_1$ implies $h[n] \in l_2$. To see this, note that if $h[n] \in l_1$, $h[n]$ can only have a finite number of positions, call them $\mathcal{K}$, so that $|h[k]| > 1$ if $k \in \mathcal{K}$. For those positions the sum is still finite if we take the square, and for the remaining positions the sum is decreased when we take the square.

(iii) $A = -2e^{-18}$.

(iv) $y[n]$ is 1 for $n < 0$, 0 for $n = 0, 1, 2, 3$, and -1 for $n > 3$.

(v) H is a set of functions which are piecewise-constant between $i \le x \le i+1$ for $i = 0, \cdots, N$ and otherwise equal to zero.

Since P spans H, we only need to show that its elements are orthogonal and have norm 1.

$$\int_{-\infty}^{\infty} p_i(x) p_j^*(x) \mathrm{d}x = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

(vi) Let $y = y_w + y_o$ where $y_w \in W$ and y_o is orthogonal to y_p. Due to projection theorem, y_w must be the projection of y on W in order to split y to such vectors.

Since u_1 and u_2 are orthogonal, they can use as elements of a basis for W. Let

$$e_1 = \tfrac{u_1}{|u_1|} = \tfrac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}, \; e_2 = \tfrac{u_2}{|u_2|} = \tfrac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Then $y_w = \langle y, e_1 \rangle e_1 + \langle y, e_2 \rangle e_2 = (-0.4, 2, 0.2)$ and $y_o = (1.4, 0, 2.8)$.

(vii) Note that $R_Y[k] = \delta[k]$. This corresponds to a power spectral density $P_Y(e^{j2\pi f}) = 1$. On the other hand, since $Y[n]$ is the output of the LTI system,

$$\left| H(e^{j2\pi f}) \right|^2 P_X(e^{j2\pi f}) = P_Y(e^{j2\pi f}).$$

Thus $\left| H(e^{j2\pi f}) \right|^2 = \frac{1}{P_X(e^{j2\pi f})}$. Such a filter is called a *whitening* filter since in the spectrum is flat.

Problem 2 (Inverse of Z Transform). **Name:** ____________________

Consider the signal $x[n]$ with the z-transform

$$X(z) = \frac{72 - 48z^{-1} + 18z^{-2} + z^{-3}}{4((1 - \frac{1}{2}z^{-1})^2(18 - 6z^{-1} + 2z^{-2}))},$$

and the ROC $|z| > a$.

(i) [5 pts] Find the value of a?

(ii) [5 pts] Does $\sum_{n=-\infty}^{\infty} x[n]$ converge? If so, find the limit?

(iii) [15 pts] Compute $x[0]$ and $x[1]$.

Solution:

(i) The poles are at $1/2$ and $1/3$ so $a = 1/2$.

(ii) Yes, the limit is $X(1) = \frac{43}{14} = 3.07143$.

(iii) $x[0] = \lim_{z\to\infty} X(z) = 1$ and $x[1] = \lim_{z\to\infty} z(X(z) - x[0]) = 2/3$.

Problem 3 (Yet Another Filter Design Method)**.** [30 pts] **Name:** __________________

Consider a symmetric and real valued FIR filter with z-transform $H(z) = \sum_{n=-N}^{N} h[n]z^{-n}$, where $h[n] = h[-n]$.

We learned in class two basic filter design methods, namely the window method as well as the min-max method. Here is a third possibility.

We are given the desired ideal filter, call it $H_{\text{ideal}}(e^{2\pi jf})$. Pick K frequency points $\{f_k\}_{k=1}^{K}$, $0 \le f_1 < f_2 < \cdots < f_K \le \frac{1}{2}$. Design the filter $H_{\text{ideal}}(e^{2\pi jf})$ so that for $1 \le k \le K$

$$H(e^{2\pi jf_k}) = H_{\text{ideal}}(e^{2\pi jf_k}).$$

Explain how you can efficiently design a filter according to this design criteria. What are valid choices for K?

Solution: Here is the procedure. Set $x = \cos(2\pi f)$ and use Chebyshev polynomials to rewrite the filter in the form of a polynomial, call it $p(x) = \sum_{n=1}^{N} p_n x^n$ of degree at most N. In the same manner, the ideal filter is now transformed to a function, call it $D(x)$, $x \in [-1, 1]$, and the frequencies f_k are transformed to the points x_k, $-1 \le x_K < \cdots < x_2 < x_1 \le 1$. Our design criterion now asks that for $1 \le k \le K$,

$$p(x_k) = D_k = D(x_k).$$

This can be written in matrix form as

$$\begin{pmatrix} x_1^0 & x_1^1 & \ldots & x_1^N \\ x_2^0 & x_2^1 & \ldots & x_2^N \\ \ldots & \ldots & \ldots & \ldots \\ x_K^0 & x_K^1 & \ldots & x_k^N \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \ldots \\ p_N \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \\ \ldots \\ p_K \end{pmatrix}$$

The matrix is a Vandermonde matrix and the system of equations has a solution as long as $K \le N + 1$.

Problem 4 (Zero Padding and Interleaving). **Name:** ______________________________

Let $x[n]$ be a discrete-time signal of length N and let $X[k]$ denote its DFT. Let $\alpha_N = e^{-\frac{2\pi j}{N}}$. Let $x_i[n]$, $i = 1, 2, 3$, denote discrete-time signal of length $2N$ and let $X_i[k]$ denote their DFTs. We have

$$x_1[n] = \begin{cases} x[n], & 0 \le n < N, \\ 0, & N \le n < 2N. \end{cases}$$

$$x_2[n] = \begin{cases} x[n/2], & 0 \le n < 2N, n \text{ is even} \\ 0, & \text{else.} \end{cases}$$

$$x_3[n] = \begin{cases} x[n], & 0 \le n < N, \\ x[n-N], & N \le n < 2N. \end{cases}$$

(i) [10 pts] Express $X_1[2k]$ in terms of $X[k]$.

(ii) [10 pts] Express $X_2[k]$ in terms of $X[k]$.

(iii) [10 pts] Express $X_3[k]$ in terms of $X[k]$.

Solutions:

(i) We have $X_1[2k] = \sum_{n=0}^{2N-1} x_1[n]\alpha_{2N}^{n2k} = \sum_{n=0}^{N-1} x[n]\alpha_{2N}^{n2k} = \sum_{n=0}^{N-1} x[n]\alpha_N^{nk} = X[k]$.

(ii) We have $X_2[k] = \sum_{n=0}^{2N-1} x_2[n]\alpha_{2N}^{nk} = \sum_{n=0}^{N-1} x[n]\alpha_N^{nk} = X[k \bmod N]$.

(iii) We have $X_3[k] = \sum_{n=0}^{2N-1} x_3[n]\alpha_{2N}^{nk} = \sum_{n=0}^{N-1} x[n](\alpha_{2N}^{nk} + \alpha_{2N}^{(n+N)k})$. The last sum is 0 for k odd and $2X[k/2]$ for k even.