ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 20 Signal Processing for Communications
Solution 10 May 9, 2011,INF 213 – 10:15am-12:00

Problem 1.

1)

$$\begin{aligned}
\int x(t)e^{-j2\pi ft}dt &= X(f) \\
\Rightarrow \int x(t-\tau)e^{-j2\pi ft}dt &= e^{-j2\pi f\tau}\int x(\tilde{t})e^{-j2\pi f\tilde{t}}d\tilde{t} \\
&= e^{-j2\pi f\tau}X(f)
\end{aligned}$$

2)

$$\begin{aligned}
\int x(\tau)e^{-j2\pi t\tau}d\tau &= X(t) \\
\int X(t)e^{-j2\pi ft}dt &= \int\int x(\tau)e^{-j2\pi t\tau}e^{-j2\pi ft}dtd\tau \\
&= \int x(\tau)\int e^{-j2\pi t(\tau+f)}dtd\tau \\
&= \int x(\tau)\delta(\tau+f)d\tau \\
&= x(-f)
\end{aligned}$$

3)

$$\begin{aligned}
\int x(at)e^{-j2\pi ft}dt &= \frac{1}{a}\int x(\tilde{t})e^{\left(\frac{-j2\pi f\tilde{t}}{a}\right)}d\tilde{t} \\
&= \frac{1}{a}X\left(\frac{f}{a}\right)
\end{aligned}$$

4)

$$\begin{aligned}
y(t) &= \int x_1(\tau)x_2(t-\tau)d\tau \\
\Rightarrow \int y(t)e^{-j2\pi ft}dt &= \int\int x_1(\tau)x_2(t-\tau)e^{-j2\pi ft}d\tau dt \\
&= \int x_1(\tau)\int x_2(t-\tau)e^{-j2\pi ft}dtd\tau \\
&= \int x_1(\tau)X_2(f)e^{-j2\pi f\tau}d\tau \\
&= X_2(f)\int x_1(\tau)e^{-j2\pi f\tau}d\tau = X_1(f)X_2(f)
\end{aligned}$$

Problem 2.

$$\begin{aligned} X_s(f) &= X(f) * P(f) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T_s}\right) \end{aligned}$$

If $x(t)$ is bandlimited to $\frac{1}{2T_s}$, then no aliasing occurs in the above sum as $f_N = \frac{1}{2T_s} \leq \frac{f_s}{2}$.

Problem 3. Let $x(t) = e^{j2\pi f_0 t}$, where $f_0 = 10KHz$. Then the sampled version would be $x[n] = e^{j\omega_0 n}$, where $\omega_0 = 2\pi \frac{f_0}{F_s}$ and $F_s = 8KHz$. So in this example $x[n] = e^{j2\pi f_b}$ with $f_b = 2KHz$ and in fact all continuous-time frequencies of the form $f = (2{+}8k) \times 10^3 Hz (f_b = 2000Hz < 4000Hz = \frac{F_s}{2})$ are aliased to the same discrete-time frequency $f_b = 2KHz$ which is thus the perceived frequency of the interpolated sinusoid.

Problem 4. 1) What we know from $x(t)$ is that it is time limited. This says that $x(t)$ is not bandlimited in frequency domain. Now if we sample this signal in any desired sampling frequency F_s, we cannot avoid aliasing due to the non-zero $X(f)$ in the whole spectrum. (Look at figure 9.12 of the textbook).

2) The Fourier transform of a continuous-time signal $x(t)$ and its inversion formula are defined as 9.4 and 9.5 in the textbook but their convergence is only assured for functions which satisfy the so-called Dirichlet conditions. In particular, the FT is always well defined for square integrable (finite energy), continuous-time signals. Let's first check if $x_c(t)$ is a finite energy :

$$\int_{-\infty}^{\infty} e^{-2\frac{t}{T_s}} dt = -\frac{T_s}{2}(e^{-\infty} - e^{\infty}) = \infty$$

So the following step CANNOT be written and concluded for $x(t) = e^{-\frac{t}{T_s}}$:

$$\begin{aligned} X(j\Omega) &= \left(\frac{\pi}{f_N}\right) rect\left(\frac{\pi}{2f_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi(f/f_N)n} \\ &= \begin{cases} \frac{\pi}{f_N} X(e^{j\pi f/f_N}) & \text{for } |f| \leq f_N \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Problem 5. Note that the solution is given in terms of the angular frequency!

1) Bandwith of the system is $2\Omega_0$.Then minimum sampling period becomes $\frac{\pi}{2\Omega_0}$.

2) Figure 1 shows the DTFT of the signal of the sampled signal $x_a[n]$ with a sampling period $\frac{\pi}{\Omega_0}$.

3) First we convert discrete signal to continuous time. Then we filter the signal in order to get one side lobe, and we shift the corresponding signal (cos modulation). If you repeat the same procedure for the other side lobe, we get the original signal.

4) When we shift in frequency domain in order to prevent aliasing we need to be sure that the baseband $2\Omega_0$ is large enough. That is :

$$\begin{aligned} 2(\Omega_1 - \Omega_0) &\leq 2\Omega_0 \\ = \Omega_1 &\leq 2\Omega_0 \end{aligned}$$

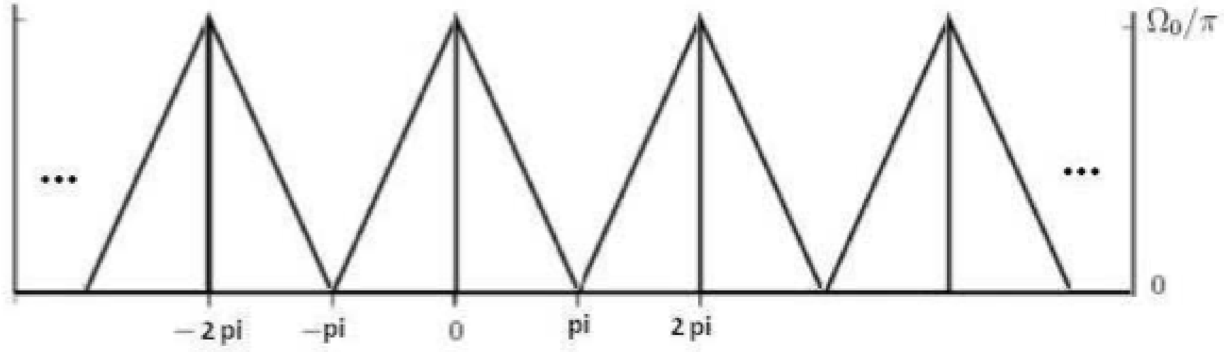

Figure 1: DTFT of discrete signal $x_a[n]$

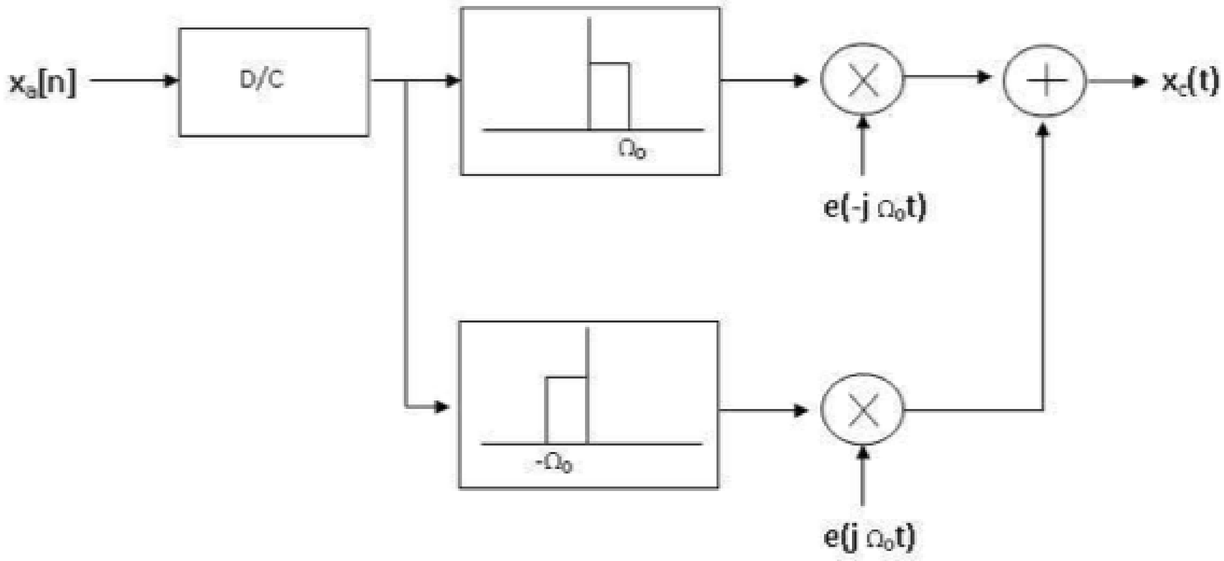

Figure 2: The reconstruction of $x_c(t)$

If the baseband $2\Omega_0$ is not large enough or the bandwidth $2(\Omega_1 - \Omega_0)$ is not small enough, then the smallest sampling frequency becomes Nyquist frequency.

$$\Omega_s = \begin{cases} 2(\Omega_1 - \Omega_0) & \text{if } \Omega_1 \leq 2\Omega_0 \\ 2\Omega_1 & \text{else.} \end{cases}$$