ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 14 Signal Processing for Communications

Solution 7 April 4, 2011,INF 213 – 10:15am-12:00

Problem 1 (The world of Ideals)**.** We want to derive a relation between $X(e^{j2\pi f})$ and the DTFT of the downsampled signal, i.e $X_d(e^{j2\pi f})$. We do this in two steps.
Step 1 : Consider first the signal

$$x_p[n] = \begin{cases} x[n] & n = 4k, k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Now, observe that

$$\begin{aligned} x_p[n] &= \frac{1}{4}\Big(x[n] + (e^{j\frac{2\pi}{4}})^n x[n] + (e^{j\frac{2\pi}{4}2})^n x[n] + (e^{j\frac{2\pi}{4}3})^n x[n]\Big) \\ &= \frac{1}{4}\sum_{k=0}^{3} e^{j\frac{2\pi}{4}kn} x[n] \end{aligned}$$

Note that the complex exponentials are periodic with period 4, so it is sufficient to check that the above relation holds for $n = 0, 1, 2, 3$.
Using the frequency shift property of the DTFT, we get

$$X_p(e^{j2\pi f}) = \frac{1}{4}\sum_{k=0}^{3} X(e^{j(2\pi f - \frac{2\pi}{4}k)})$$

Step 2 : Removing all zeros introduced in $x_p[n]$, we get $x_d[n]$. Hence,

$$x_d[n] = x_p[4n]$$

So,

$$\begin{aligned} X_d(e^{j2\pi f}) &= \sum_{n\in\mathbb{Z}} x_d[n] e^{-j2\pi f n} \\ &= \sum_{n\in\mathbb{Z}} x_p[4n] e^{-j2\pi f n} \\ &\underset{m=4n}{=} \sum_{m=4n, n\in\mathbb{Z}} x_p[m] e^{-j2\pi f \frac{m}{4}} \\ &\underset{(a)}{=} \sum_{m\in\mathbb{Z}} x_p[m] e^{-j2\pi f \frac{m}{4}} \\ &= X_p(e^{j\frac{2\pi f}{4}}) \end{aligned}$$

Where (a) follows from the fact that $x_p[m] = 0$, for $m \neq$ integer multiple of 4.
So overall, we obtained

$$X_d(e^{j2\pi f}) = \frac{1}{4}\sum_{t=0}^{3} X(e^{j(\frac{2\pi f}{4} - \frac{2\pi}{4}k)})$$

Now, we want to derive a relationship between $X_d(e^{j2\pi f})$ and the DTFT of the upsampled signal, i.e $X_u(e^{j2\pi f})$.

$$\begin{aligned}
X_u(e^{j2\pi f}) &= \sum_{n\in\mathbb{Z}} x_u[n]e^{-j2\pi fn} \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} x_d[k]\delta[n-4k]\right) e^{-j2\pi fn} \\
&= \sum_{k\in\mathbb{Z}} x_d[k] \sum_{n\in\mathbb{Z}} \delta[n-4k]e^{-j2\pi fn} \\
&= \sum_{k\in\mathbb{Z}} x_d[k]e^{-j2\pi f4k} \\
&= X_d(e^{j2\pi f4})
\end{aligned}$$

The cascade of downsampling and upsampling operations yields:

$$X_u(e^{j2\pi f}) = \frac{1}{4}\sum_{k=0}^{3} X(e^{j(2\pi f-\frac{2\pi}{4}k)})$$

The signals are drawn in figures 1-5 at the end of the solutions.

Problem 2 (Fractional Delay)**.**

a) The system represents a "fractional" delay. Hence,

$$y[n] = \sin\left(2\pi f_0(n-d)+\phi_0\right)$$

i.e. $y[n]$ is "delayed" by d time units. The simplest way to compute $y[n]$ is to transform into the Fourier domain, to multiply, and to transform back.

b) We have $h[n] = \delta[n-d]$. Since this impulse response is absolutely summable, the system is BIBO stable. If $d \geq 0$ then the system is causal, otherwise it is not.

c) If d is not an integer, then we get by direct integration :

$$\begin{aligned}
h[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi fd}e^{j2\pi fn} \\
&= \frac{1}{2j\pi(n-d)}e^{j2\pi(n-d)}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \mathrm{sinc}(n-d)
\end{aligned}$$

In this case the impulse response is not absolutely summable, so the system is not BIBO stable. The system is never causal.

Problem 3.

$$H(z) = \frac{(1-\frac{1}{3}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})}$$

1)

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{1-\frac{1}{3}z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}+\frac{1}{4}z^{-3}} \\ &= \frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-3}} \end{aligned}$$

The difference equation is :

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{4}y[n-3] = x[n] - \frac{1}{3}x[n-1]$$

2) Note that the poles are located on the circles $|z| = \frac{1}{2}$ and $|z| = \frac{1}{\sqrt{2}}$. See Fig. 6.
We can associate three regions with H(z).
$\text{ROC}_1 : |z| > \frac{1}{\sqrt{2}}$. See Fig. 7.
$\text{ROC}_2 : |z| < \frac{1}{2}$. See Fig 8.
$\text{ROC}_3 : \frac{1}{2} < |z| < \frac{1}{\sqrt{2}}$. See Fig 9.
ROC_1 includes the unit circle, so it is a stable system. Moreover, ROC_1 extends outward from $|z| = \frac{1}{\sqrt{2}}$ including ∞. Hence it is a causal system.
On the other hand ROC_2 does not include the unit circle, hence it is not stable. Since ROC_2 is the inside of a circle, it cannot be causal either. Similarly, ROC_3 is neither stable, nor causal.

3) The Fourier transform converges in ROC_1, see Fig. 10.

4)

$$H(z) = \frac{(1-\frac{1}{3}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})}$$

ROC_1:$|z| > \frac{1}{\sqrt{2}}$

$$\frac{(1-\frac{1}{3}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})} = \frac{A}{(1+\frac{1}{2}z^{-1})} + \frac{B+Cz^{-1}}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$

$$H(z)(1+\frac{1}{2}z^{-1})|_{z=-\frac{1}{2}} = \frac{1-\frac{1}{3}(-2)}{1-(-2)+\frac{1}{2}(-2)^2} = \frac{1}{3}$$

$$\begin{aligned} 1-\frac{1}{3}z^{-1} &= A\left(1-z^{-1}+\frac{1}{2}z^{-2}\right) + \left(1+\frac{1}{2}z^{-1}\right)(B+Cz^{-1}) \\ &= (A+B) + \left(-A+C+\frac{B}{2}\right)z^{-1} + \left(\frac{A}{2}+\frac{C}{2}z^{-2}\right) \end{aligned}$$

$\Rightarrow B = \frac{2}{3}$
$\Rightarrow C = -\frac{1}{3}$
Hence, the partial fraction expansion gives :

$$H(z) = \frac{\frac{1}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{\frac{2}{3}\left(1 - \frac{1}{2}z^{-1}\right)}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Since $H(z)$ is causal inside ROC_1, the inverse is given by :

$$h[n] = \frac{1}{3}\left(\frac{-1}{2}\right)^n u[n] + \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right) u[n]$$

ROC_2: $|z| < \frac{1}{2}$
You need further to expand the second fraction with partial fraction expansion.

$$\frac{\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{D}{\left(1 - \frac{\sqrt{2}}{2}e^{\left(j\frac{\pi}{4}\right)}z^{-1}\right)} + \frac{E}{\left(1 - \frac{\sqrt{2}}{2}e^{\left(-j\frac{\pi}{4}\right)}z^{-1}\right)}$$

$\Rightarrow D = \frac{1}{2}$
$\Rightarrow E = \frac{1}{2}$
So

$$H(z) = \frac{\frac{1}{3}}{\left(1 + \frac{1}{2}z^{-1}\right)} + \frac{\frac{2}{3}D}{\left(1 - \frac{\sqrt{2}}{2}e^{\left(j\frac{\pi}{4}\right)}z^{-1}\right)} + \frac{\frac{2}{3}E}{\left(1 - \frac{\sqrt{2}}{2}e^{\left(-j\frac{\pi}{4}\right)}z^{-1}\right)}$$

Since $H(z)$ is anti-causal inside ROC_2, the inverse is given by:

$$\begin{aligned} h[n] &= -\frac{1}{3}\left(\frac{-1}{2}\right)^n u[-n-1] - \frac{2}{3}D\left(\frac{\sqrt{2}}{2}e^{\left(j\frac{\pi}{4}\right)}\right)^n u[-n-1] - \frac{2}{3}E\left(\frac{\sqrt{2}}{2}e^{\left(-j\frac{\pi}{4}\right)}\right)^n u[-n-1] \\ &= -\frac{1}{3}\left(\frac{-1}{2}\right)^n u[-n-1] - \frac{2}{3}Re\left\{\left(\frac{\sqrt{2}}{2}e^{\left(j\frac{\pi}{4}\right)}\right)^n\right\} u[-n-1] \end{aligned}$$

ROC_3: $\frac{1}{2} < |z| < \frac{1}{\sqrt{2}}$
Using the previous partial fraction expansion, we see that the first fraction of $H(z)$ is causal, the other two fractions are anti-causal. Hence,

$$h[n] = \frac{1}{3}\left(\frac{-1}{2}\right)^n u[n] - \frac{2}{3}Re\left\{\left(\frac{\sqrt{2}}{2}e^{\left(j\frac{\pi}{4}\right)}\right)^n\right\} u[-n-1]$$

5) We want to find $G(z)$ such that $H(z)G(z) = 1$

i)

$$G(z) = \frac{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)}$$

See Fig. 11. We found that $H(z)$ is both stable and causal inside ROC_1. Similarly we can deduce that $G(z)$ is both causal and stable for the region ROC_G: $|z| > \frac{1}{3}$. Now, we also need to ensure that $\text{ROC}_1 \cap \text{ROC}_G \neq \varnothing$, which holds in this case. The DTFT is plotted in Fig. 12.

ii)

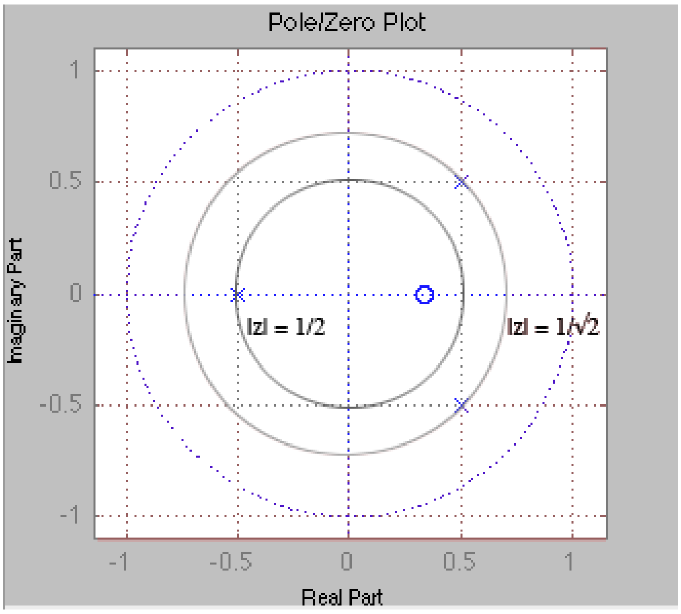

Figure 6: Pole-zero plot

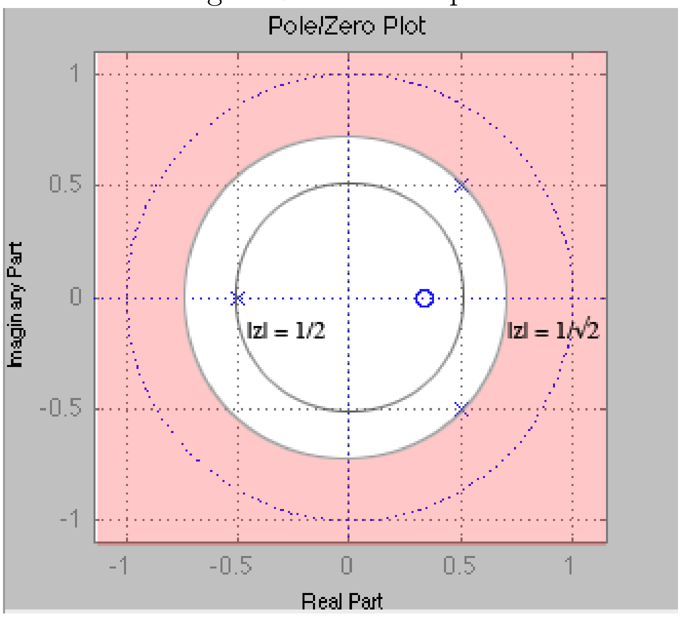

Figure 7: $\text{ROC}_1 : |z| > \frac{1}{\sqrt{2}}$

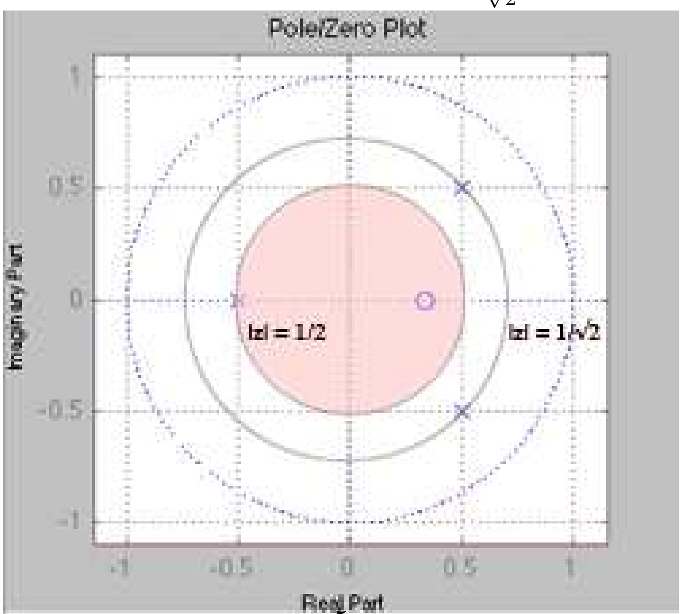

Figure 8: $\text{ROC}_2 : |z| < \frac{1}{2}$

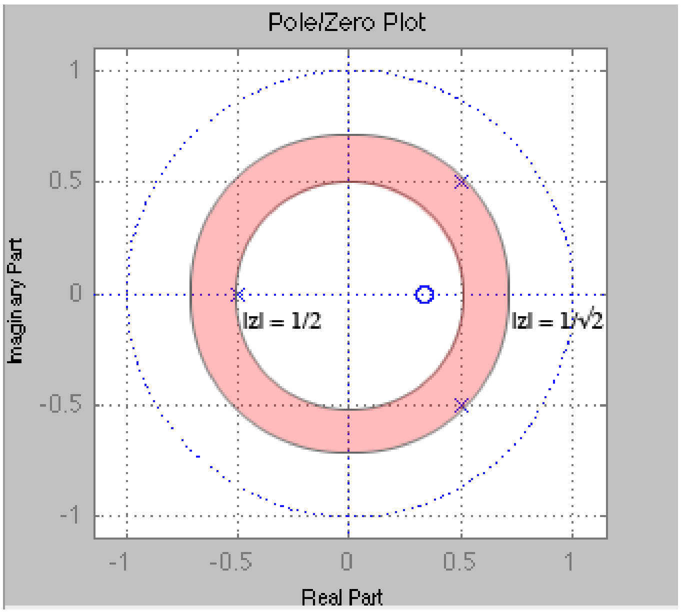

Figure 9: $\mathrm{ROC}_3 : \frac{1}{2} < |z| < \frac{1}{\sqrt(2)}$

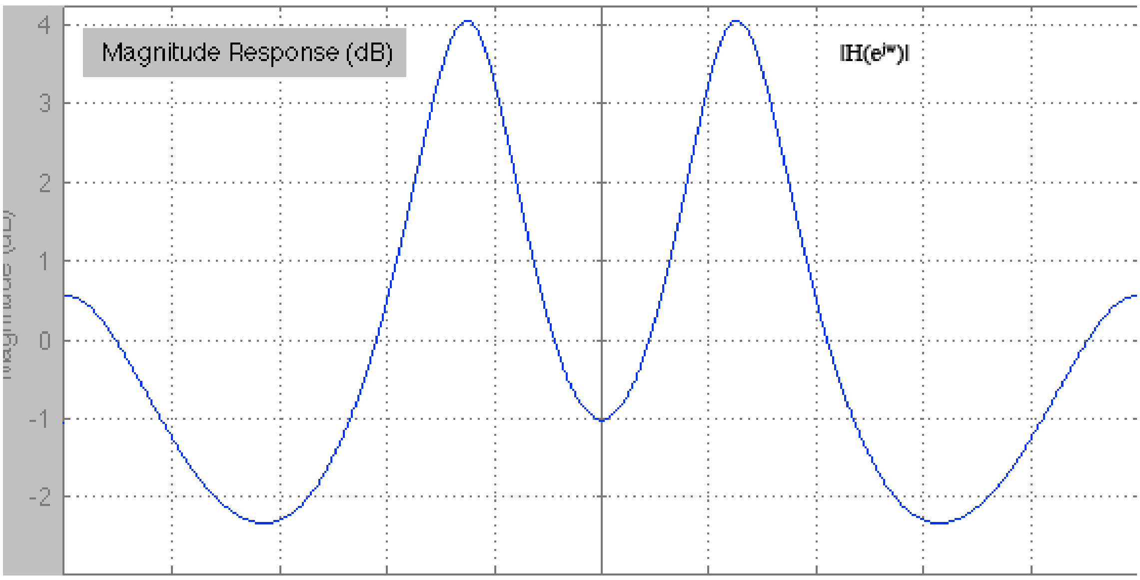

Figure 10: Magnitude response of $|H(e^{j2\pi f})|$

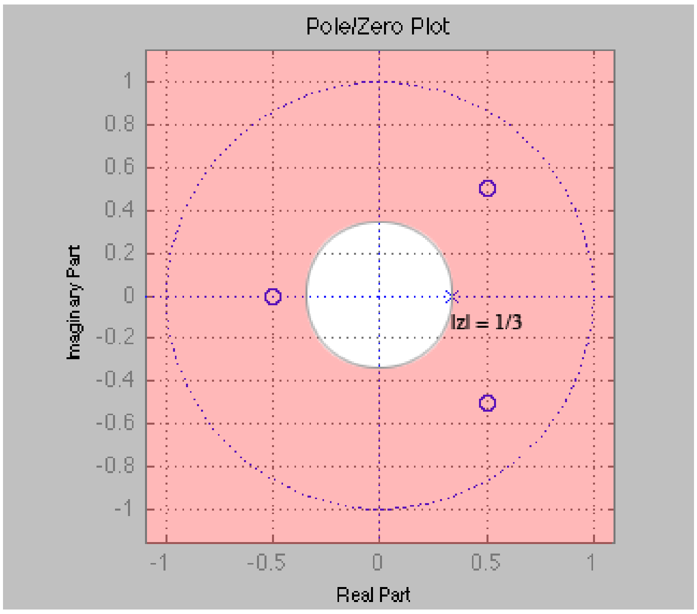

Figure 11: Pole-zero plot with ROC for $G(z)$

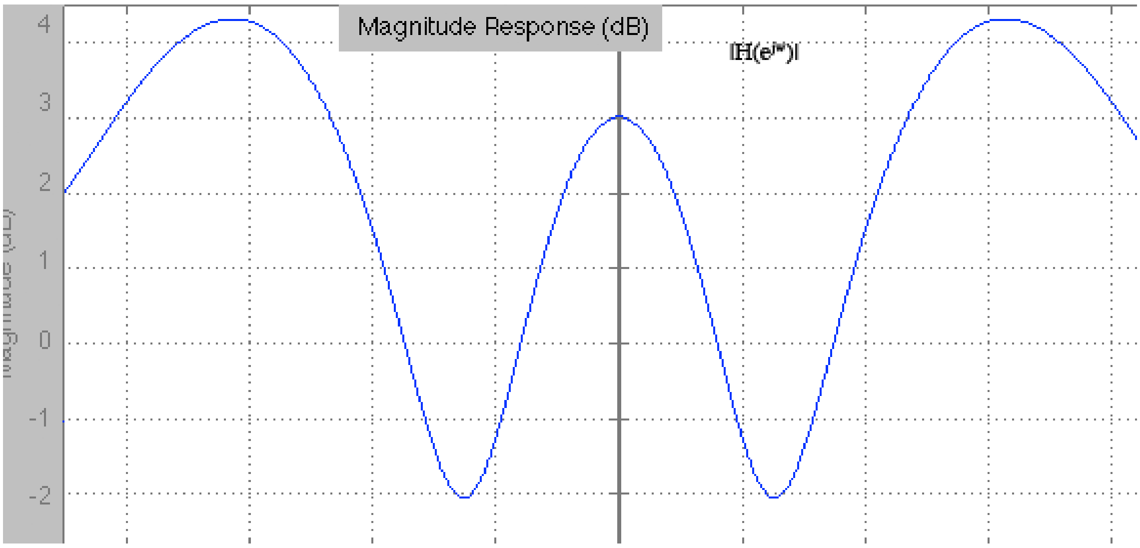

Figure 12: Magnitude response for $G(z)$

Problem 4. (FIR Approximation of the Hilbert Filter/Oppenheim Problems 7.32/7.33/7.52)

1) i)

$$\begin{aligned}
H_s(e^{j2\pi f}) &= \sum_{n=0}^{M} h_s[n]e^{-j2\pi fn} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h_s[n]e^{-j2\pi fn} + \sum_{n=\frac{M+1}{2}}^{M} h_s[n]e^{-j2\pi fn} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h_s[n]e^{-j2\pi fn} + \sum_{m=0}^{\frac{M-1}{2}} h_s[M-m]e^{-j2\pi f(M-m)} \\
&= e^{-j2\pi f\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} h_s[n]e^{j2\pi f(\frac{M}{2}-n)} + \sum_{n=0}^{\frac{M-1}{2}} h_s[n]e^{-j2\pi f(\frac{M}{2}-n)} \\
&= e^{-j2\pi f\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} 2h_s[n]\cos\left(2\pi f\left(\frac{M}{2}-n\right)\right) \\
&= e^{-j2\pi f\frac{M}{2}} \sum_{n=1}^{\frac{M+1}{2}} 2h_s\left[\frac{M+1}{2}-n\right]\cos\left(2\pi f\left(n-\frac{1}{2}\right)\right)
\end{aligned}$$

ii) Similarly, by considering the fact that $h[n] = 0$ for $n < 0$ and $n > M$, and the fact that h[n] = h[M-n] for $n = 0, ..., \frac{M-1}{2}$, we can derive:

$$H_{as}(e^{j2\pi f}) = je^{j2\pi f\frac{M}{2}} \sum_{k=1}^{\frac{M+1}{2}} 2h_{as}\left[\frac{M+1}{2}-k\right]\sin\left(2\pi f\left(k-\frac{1}{2}\right)\right)$$

2) i) The Hilbert transform is given by:

$$H_h(e^{j2\pi f}) = \begin{cases} e^{j\frac{\pi}{2}} & -\frac{1}{2} < f < 0 \\ e^{-j\frac{\pi}{2}} & 0 < f < \frac{1}{2} \end{cases}$$

Hence, the delayed Hilbert transform with generalized linear phase can be defined as:

$$H_d(e^{j2\pi f}) = \begin{cases} e^{j\frac{\pi}{2}-j2\pi fd} & -\frac{1}{2} < f < 0 \\ e^{-j\frac{\pi}{2}-j2\pi fd} & 0 < f < \frac{1}{2} \end{cases}$$

The magnitude and phase responses are plotted in Fig. 13, and Fig. 14 respectively.

ii) Note that the phase response has a π radian phase shift at $f = 0$. This is because the above Hilbert filter requires a zero at $z = 1$. This implies that the filter coefficients sum up to 0. Hence the filter should be antisymmetric. So we could only use $h_{as}[n]$ to approximate $h_d[n]$.

iii)

$$
\begin{aligned}
h_d[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} H_d(e^{j2\pi f})e^{j2\pi fn}df \\
&= \int_{-\frac{1}{2}}^{0} e^{j\left(\frac{\pi}{2}-2\pi fd\right)}e^{j2\pi fn}df + \int_{0}^{\frac{1}{2}} e^{-j\left(\frac{\pi}{2}+2\pi fd\right)}e^{j2\pi fn}df \\
&= e^{j\frac{\pi}{2}}\int_{-\frac{1}{2}}^{0} e^{j2\pi f(n-d)}df + e^{-j\frac{\pi}{2}}\int_{0}^{\frac{1}{2}} e^{j2\pi f(n-d)}df \\
&= \begin{cases} \dfrac{1}{\pi(n-d)}\left[1+\sin\left(\pi(n-d)-\frac{\pi}{2}\right)\right] & n \neq d \\ 0 & n = d \end{cases} \\
&= \begin{cases} \dfrac{1}{\pi(n-d)}\left[1-\cos\left(\pi(n-d)\right)\right] & n \neq d \\ 0 & n = d \end{cases} \\
&= \begin{cases} \dfrac{2\sin^2\left(\frac{\pi}{2}(n-d)\right)}{\pi(n-d)} & n \neq d \\ 0 & n = d \end{cases}
\end{aligned}
$$

We observe that $h_d[n]$ is symmetric around $n = d$. Moreover, from part (ii) we also know that $h_{as}[n]$ can be used to approximate $h_d[n]$ as a causal, FIR filter with generalized linear phase. Hence $d = \frac{M}{2}$ since $\frac{M}{2}$ is the axis of symmetry of $h_{as}[n]$.

iv) From Parseval, we have

$$
\epsilon^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|H(e^{j2\pi f}) - H_d(e^{j2\pi f})\right|^2 df = \sum_{n\in\mathbb{Z}} |h_d[n] - h[n]|^2
$$

We know that due to the windowing operation $h[n] = 0$ for $n < 0$ and $n > M$. As a result to minimize ϵ^2, the best thing we could do is to select $h[n] = h_d[n]$ for $0 \leq n \leq M$. Therefore the optimal window which minimizes ϵ^2 is the rectangular window, i.e :

$$
w[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}
$$

Problem 5. Note that the solution is given in terms of the "w" variable notation for DTFT!

$$\begin{aligned}
z[n] &= x[n]*y[n],\ z[n] = \frac{1}{2\pi}\int_0^{2\pi} 2e^{j\omega}e^{j\omega n}d\omega \\
\Rightarrow &\quad Z(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega}).
\end{aligned}$$

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{2}2\pi\left[e^{j\frac{\pi}{17}}\delta\left(\omega-\frac{\pi}{49}\right)+e^{-j\frac{\pi}{17}}\delta\left(\omega+\frac{\pi}{49}\right)\right] \\
Y(e^{j\omega}) &= \sum_{n=-49}^{49}\frac{(49-n)}{49}e^{-j\omega n} \\
z[n] &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-49}^{49}\frac{(49-n')}{49}e^{-j\omega n'}\right)\frac{1}{2}2\pi\left[e^{j\frac{\pi}{17}}\delta\left(\omega-\frac{\pi}{49}\right)+e^{-j\frac{\pi}{17}}\delta\left(\omega+\frac{\pi}{49}\right)\right]e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\frac{1}{2}e^{j\frac{\pi}{17}}\left(\sum_{n'=-49}^{49}\frac{(49-n')}{49}e^{j\frac{\pi}{49}n'}\right)e^{j\frac{\pi}{49}n}2\pi+\frac{1}{2\pi}\frac{1}{2}e^{-j\frac{\pi}{17}}\left(\sum_{n'=-49}^{49}\frac{(49-n')}{49}e^{j\frac{\pi}{49}n'}\right)e^{-j\frac{\pi}{49}n}2\pi
\end{aligned}$$

Let a be $\sum_{n'=-49}^{49}\frac{49-n'}{49}e^{-j\frac{\pi}{49}n'}$ and a* be $\sum_{n'=-49}^{49}\frac{49-n'}{49}e^{j\frac{\pi}{49}n'}$. Then we have :

$$z[n] = \frac{1}{2}e^{j\frac{\pi}{17}}ae^{j\frac{\pi}{49}n}+\frac{1}{2}e^{j\frac{\pi}{17}}a^*e^{-\frac{\pi}{49}n}$$

Now we only need to compute a.

$$a = \sum_{n'=0}^{98}\frac{n'}{49}e^{-j\frac{\pi}{49}(-n'+49)} = \frac{e^{-j\pi}}{49}\sum_{n'=0}^{98}n'e^{j\frac{\pi}{49}n'}$$

We know from hw1 that

$$\begin{aligned}
\sum_{n=0}^{k}\alpha^n &= \frac{1-\alpha^{k+1}}{1-\alpha} \\
\sum_{k=1}^{n}kx^k &= x\sum_{k=1}^{n}kx^{k-1} \\
&= x\sum_{k=1}^{n}\frac{dx^k}{dx} \\
&= x\frac{d}{dx}\left(\sum_{k=1}^{n}x^k\right) \text{ by linearity of differentiation} \\
&= x\frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}-1\right) \\
&= x\frac{-(n+1)x^n(1-x)-(1-x^{n+1})(-1)}{(1-x)^2} \\
&= \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}
\end{aligned}$$

So,

$$a = \frac{e^{-j\pi}}{49}\frac{e^{j\frac{\pi}{49}}-(99)e^{j\frac{\pi}{49}99}+98e^{j\frac{\pi}{49}100}}{(1-e^{j\frac{\pi}{49}})^2}$$

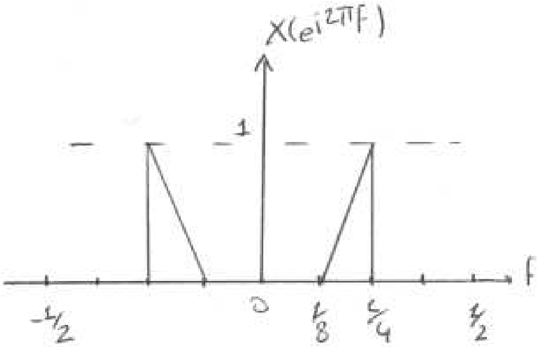

Figure 1: Original DTFT

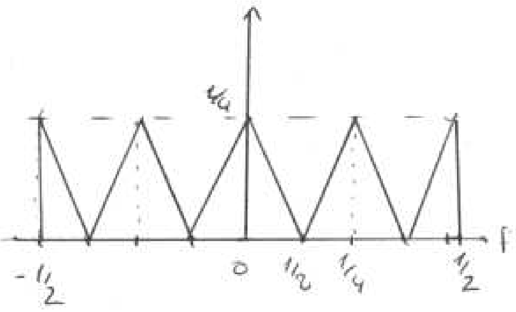

Figure 2: DTFT After Up and Down Sampling

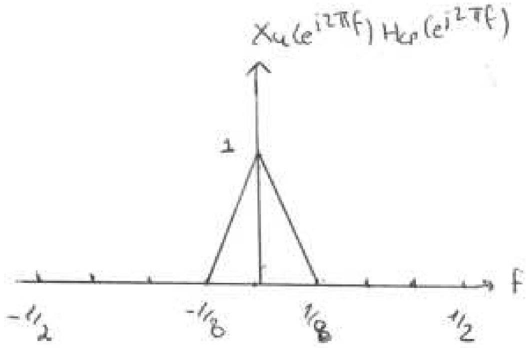

Figure 3: DTFT Using the Low Pass Filter

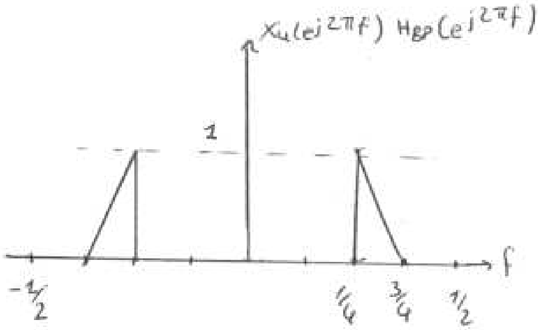

Figure 4: DTFT Using the Band Pass Filter

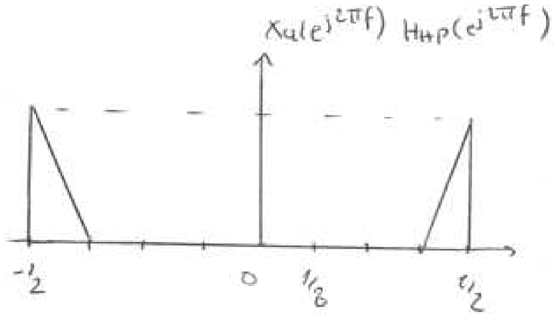

Figure 5: DTFT Using the High Pass Filter

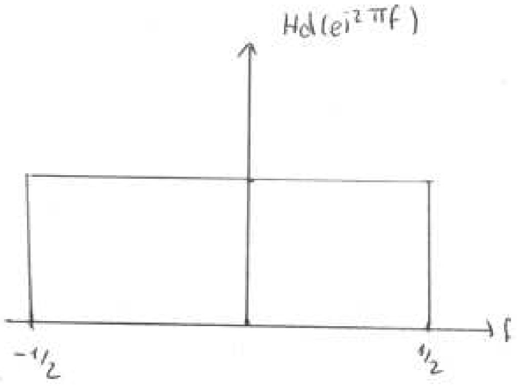

Figure 13: Magnitude Response of the filter

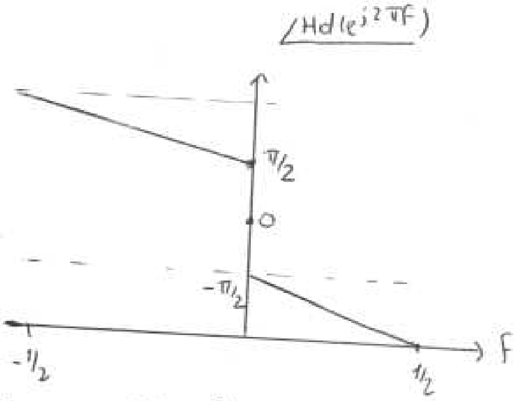

Figure 14: Phase Response of the Filter